Multi-moment maps for special Ricci-flat metrics

Andrew Swann

Department of Mathematics, Centre for Quantum Geometry of Moduli Spaces,
and DIGIT, University of Aarhus
swann@math.au.dk

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Outline

1. Ricci-flat special holonomy

2. Multi-Hamiltonian torus actions

3. Singular orbits and topological quotients
   - Flat models
   - General

4. Realisation via multi-moment maps
   - Flat model
   - General case
The Berger holonomy classification 1955,…, has only the following non-trivial irreducible Ricci-flat geometries

<table>
<thead>
<tr>
<th>Name</th>
<th>Group</th>
<th>Dimension</th>
<th>Form degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calabi-Yau</td>
<td>SU($n$)</td>
<td>2$n$</td>
<td>2, n, n</td>
</tr>
<tr>
<td>HyperKähler</td>
<td>Sp($n$)</td>
<td>4$n$</td>
<td>2, 2, 2</td>
</tr>
<tr>
<td>$G_2$ holonomy</td>
<td>$G_2$</td>
<td>7</td>
<td>3, 4</td>
</tr>
<tr>
<td>Spin(7) holonomy</td>
<td>Spin(7)</td>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

In the presence of symmetries, moment map techniques from symplectic geometry may be used if there is a closed form of degree 2, yielding many examples.
**Toric Calabi-Yau**

Include symplectic quotients of $\mathbb{C}^N$ by subtori of $T^N$ whose weights sum to zero.

$$\mathbb{C}^4 \sslash \text{diag}(e^{i\theta}, e^{i\theta}, e^{-i\theta}, e^{-i\theta}) = (\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{C}P(1))$$

**Hypertoric manifolds**

Include hyperKähler quotients of $\mathbb{H}^N$ by subtori of $T^N$.

$$T^*\mathbb{C}P(n) = \mathbb{H}^{n+1} \sslash e^{i\theta} \mathbb{1}_{n+1}$$
Other constructions

On the other hand there are complete special holonomy metrics not obtained in such a way. These include the first complete examples found by Bryant and Salamon (1989)

<table>
<thead>
<tr>
<th>$M^7$</th>
<th>$\Lambda^2(S^4)$</th>
<th>$\Lambda^2(\mathbb{C}P^2)$</th>
<th>$S^3 \times \mathbb{R}^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isom$_0$</td>
<td>Sp(2)</td>
<td>SU(3)</td>
<td>SU(2) $\times$ SU(2) $\times$ U(1)</td>
</tr>
<tr>
<td>rank(Isom)</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

for $G_2$, and the Spin-bundle of $S^4$, Isom$_0 = \text{SO}(5) \times U(1)$ of rank 3, for Spin(7).

Aim

Exploit forms of higher degree in such cases

Note: on compact manifolds, Ricci-flat implies that Killing vector fields are parallel and so the holonomy reduces. We will thus be interested in the non-compact situation.
(M, α) a manifold with a closed α ∈ Ω^p(M) preserved by G = T^n is *multi-Hamiltonian* if it there is a G-invariant

\[ \nu: M \to \Lambda^{p-1} g^* \cong \mathbb{R}^N, \]

\[ d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1} \rangle = \alpha(X_1, \ldots, X_{p-1}, \cdot) \]

for all \(X_i \in g\).

- For \(p = 2\) this is an ordinary symplectic moment map.
- \(\nu}\) invariant \(\iff\) \(\alpha\) pulls-back to 0 on each \(T^n\)-orbit
- \(b_1(M) = 0 \iff\) each \(T^n\)-action preserving \(\alpha\) is multi-Hamiltonian

More generally, we can consider several closed invariant forms \(\alpha_k \in \Omega^{p_k}(M)\) with multi-moment maps \(\nu_k\) and consider their product

\[ \nu = (\nu_1, \ldots, \nu_m): M \to \bigoplus_{k=1}^m \Lambda^{p_k-1} g^* \]
An interesting case is when

\[ \nu: M \to \mathbb{R}^N \]

is of full rank on the part \( M_0 \) of \( M \) where \( G = T^n \) acts freely, and

\[ N = \dim(M_0/G). \]

Then \( \nu \) locally exhibits \( M_0 \) as a principal \( T^n \)-bundle over \( U \subset \mathbb{R}^N \).

<table>
<thead>
<tr>
<th>Geometry</th>
<th>( \dim M )</th>
<th>( \deg \alpha )</th>
<th>( G )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symplectic/Kähler</td>
<td>2n</td>
<td>2</td>
<td>( T^n )</td>
</tr>
<tr>
<td>Calabi-Yau</td>
<td>2n</td>
<td>(2, ( n, n ))</td>
<td>( T^{n-1} )</td>
</tr>
<tr>
<td>HyperKähler</td>
<td>4n</td>
<td>(2, 2, 2)</td>
<td>( T^n )</td>
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<tr>
<td>( G_2 )</td>
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<td>Spin(7)</td>
<td>8</td>
<td>4</td>
<td>( T^4 )</td>
</tr>
</tbody>
</table>
Flats models

The flat symplectic/Kähler model is

- $M = \mathbb{C}^n$
- $\alpha = \omega = \sum_{k=1}^{n} dx_k \wedge dy_k = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k = \frac{i}{2} \sum_{k=1}^{n} dz_k \bar{z}_k$
- $G = T^n = \{ \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \}$
- $\nu = \mu = (\mu_1, \ldots, \mu_n)$

$$\mu_k = \frac{1}{2} |z_k|^2$$

We have

$$\mu(\mathbb{C}^n) = [0, \infty)^n$$

and $\mu$ induces a homeomorphism

$$\mathbb{C}^n / T^n \rightarrow [0, \infty)^n$$

But the latter is a manifold with corners.
Flat models, continued

For $G_2$ the flat model is

- $M = S^1 \times \mathbb{C}^3$
- $\alpha = (\varphi, \ast \varphi)$

\[
\varphi = \frac{i}{2} dx(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \text{Re}(dz_{123})
\]

\[
\ast \varphi = \text{Im}(dz_{123})dx - \frac{1}{8}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}})^2
\]

- $G = T^3 = S^1 \times T^2 = S^1 \times \{\text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_1 + \theta_2 + \theta_3 = 0\}$

For Calabi-Yau the flat model is

- $M = \mathbb{C}^n$
- $\alpha = (\omega, \text{Re} \, \Omega, \text{Im} \, \Omega)$, \quad $\omega = \frac{i}{2} \sum_{k=1}^{n} dz_{k\bar{k}}$, \quad $\Omega = dz_{12}...n$
- $G = T^{n-1} = \text{diagonal unitary matrices of determinant 1}$
Orbit spaces

For the $G_2$ case

$$M/G = (S^1 \times \mathbb{C}^3) / (S^1 \times T^2) = \mathbb{C}^3 / T^2 = \text{cone}(S^5) / T^2 = \text{cone}(S^5 / T^2)$$

And for the Calabi-Yau case

$$M/G = \mathbb{C}^n / T^{n-1} = \text{cone}(S^{2n-1} / T^{n-1})$$

$$S^{2n-1} = \left\{ (r_1 e^{it_1}, \ldots, r_{n-1} e^{it_{n-1}}) \mid r_k \geq 0 \ \forall k, \ \sum_{k=1}^{n-1} r_k^2 = 1 \right\}$$

Each $T^{n-1}$-orbit contains an element with $t_1 = t_2 = \cdots = t_{n-1}$ and that element is unique modulo $2\pi / n$ unless some $r_k$ is zero.
Thus $S^{2n-1}/T^{n-1}$ projects on to

$$\left\{(r_1^2, \ldots, r_{n-1}^2) \left| r_k \geq 0, \sum_{k=1}^{n-1} r_k^2 = 1 \right. \right\} = \Delta^{n-1} \equiv B^{n-1}$$

with fibres circles over the interior, and points over the boundary. It follows that $S^{2n-1}/T^{n-1}$ is homeomorphic to

$$\left\{(z, x) \in \mathbb{C} \times \mathbb{R}^{n-1} \left| |z|^2 + \|x\|^2 = 1 \right. \right\} = S^n$$

and $M/G = \mathbb{C}^n/T^{n-1}$ is homeomorphic to

$$\text{cone}(S^n) = \mathbb{R}^{n+1}$$
Theorem

For all the multi-Hamiltonian geometries considered, the torus actions has the property that every stabiliser is a connected subtorus. Local models around any special orbit with stabiliser $T^k$ are given by $(T^k \times \mathbb{R}^k) \times V$ where $V$ is a flat model.

For example, in the Calabi-Yau case suppose $\dim \text{Stab}_{T^{n-1}}(p) = k$. Then there are $n - 1 - k$ directions $U_1, \ldots, U_{n-1-k}$ tangent to the orbit through $p$. But $\omega$ pulls-back to 0 on the orbit, so the $U_i$ are linearly independent over $\mathbb{C}$. Now $\text{Stab}_{T^{n-1}}(p)$ is an Abelian group acting on $T_p M = \mathbb{C}^n$ as a subgroup of $\text{SU}(n)$ and fixing a $\mathbb{C}^{n-1-k}$ pointwise, so a subgroup of $\text{SU}(k+1)$. But this forces it to be a maximal torus.

Corollary

For the Calabi-Yau, hyperKähler, $G_2$ and Spin(7) cases, $M/G$ is homeomorphic to a smooth manifold.

via $\exp_p : T_p M \rightarrow M$
For $G_2$ the flat model is

- $M = S^1 \times \mathbb{C}^3$, $\alpha = (\varphi, \ast \varphi)$

$$\varphi = \frac{i}{2} dx (dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \text{Re}(dz_{123})$$

$$\ast \varphi = \text{Im}(dz_{123}) dx - \frac{1}{8} (dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}})^2$$

- $G = T^3 = S^1 \times T^2$ generators

$$U_1 = \frac{\partial}{\partial x}, \quad U_k = 2 \text{Re} \left( i \left( z_k \frac{\partial}{\partial z_k} - z_3 \frac{\partial}{\partial z_3} \right) \right), \quad k = 2, 3$$

- $\nu = (\nu_1, \nu_2, \nu_3, \nu_0)$

$$d\nu_i = \varphi(U_j, U_k, \cdot) \quad (ijk) = (123), \quad d\nu_0 = \ast \varphi(U_1, U_2, U_3, \cdot)$$

$$\nu_0 + i\nu_1 = -i z_1 z_2 z_3, \quad 2\nu_2 = |z_2|^2 - |z_3|^2, \quad 2\nu_3 = |z_3|^2 - |z_1|^2$$
Proposition

In the $G_2$ flat model, $\nu: M = S^1 \times \mathbb{C}^3 \to \mathbb{R}^4$

$$\nu_0 + i\nu_1 = -iz_1z_2z_3, \quad 2\nu_2 = |z_2|^2 - |z_3|^2, \quad 2\nu_3 = |z_3|^2 - |z_1|^2$$

induces a homeomorphism $M/G = \mathbb{C}^3/T^2 \to \mathbb{R}^4$.

This also applies to the Spin(7)-case. Similar results hold in the hyperKähler and Calabi-Yau cases.

Main point: for $t = |z_3|^2$, $c = \nu_0^2 + \nu_1^2$, satisfies $f(t) := t(t - 2\nu_3)(t + 2\nu_2) = c$

with each factor $\geq 0$.

$(t, \nu) \mapsto \nu$ is a continuous bijection

$\mathbb{R}^4 = \mathbb{C}^3/T^2 \to \mathbb{R}^5 \to \mathbb{R}^4$, so a

domain.

homeomorphism, by Brouwer’s invariance of
**Theorem**

*For multi-Hamiltonian $G_2$, Spin(7) and hyperKähler cases the multi-moment map $\nu$ induces local homeomorphisms*

$$M/G \to \mathbb{R}^N$$

Also know it holds for Calabi-Yau cases when $n \leq 3$.

**Ingredients in proof**

- properties of commuting Killing vectors at zeros
- high-order approximation by the flat model
- local understanding of image sets of singular locus
- local injectivity argument at a point
- topological degree argument combined with deformation to flat model
Commuting Killing vector fields

$X$ Killing implies

- $\nabla X$ is a skew-symmetric endomorphism of $TM$
- $\nabla^2_{A,B}X = -R_{X,A}B$

So $X_p = 0$ implies $(\nabla^2 X)_p = 0$ and $(\nabla^3 X)_p = -(R \circ \nabla X)_p$.

If $X, Y$ are Killing, commute and $X_p = 0$, then

- $\nabla X$ and $\nabla Y$ commute at $p$.

$G_2$ case, with $\text{Stab}_{T_3}(p) = T^2$, $T_p M = \mathbb{R} \oplus \mathbb{C}^3$. Can choose our generators so that $U_2, U_3$ are zero at $p$ with covariant derivatives

$$
(\nabla U_2)_p = \text{diag}(i, 0, -i), \quad (\nabla U_3)_p = \text{diag}(0, i, -i).
$$

Let $U$ be any generator that is non-zero at $p$. Then $\nabla U \in \mathfrak{g}_2$ and $\nabla U$ commutes with $\nabla U_i$, $i = 1, 2$. But rank $\mathfrak{g}_2 = 2$, so can adjust $U$ to get at $p$ $U_1$ unit length in $\mathbb{R}$ and $\nabla U_1 = 0$. 

**High-order approximation**

$G_2$ case, $\text{Stab}_{T^3}(p) = T^2$. At $p$, can ensure $\varphi$ and $\ast \varphi$ agree with the flat model,

$$U_2 = 0 = U_3, \quad \nabla U_1 = 0, \quad \nabla^2 U_2 = 0 = \nabla^2 U_3$$

and $U_1, \nabla U_2, \nabla U_3$ agree with the flat model.

Now $d\nu_i = \varphi(U_j, U_k, \cdot), (ijk) = (123)$, and $d\nu_0 = \ast \varphi(U_1, U_2, U_3, \cdot)$. But $\nabla \varphi = 0 = \nabla \ast \varphi$, so

$$\nabla^r \nu_i = \varphi(\nabla^{s_1} U_j, \nabla^{s_2} U_k, \cdot), \quad r = s_1 + s_2 + 1, \ (ijk) = (123)$$

$$\nabla^r \nu_0 = \ast \varphi(\nabla^{s_1} U_1, \nabla^{s_2} U_2, \nabla^{s_3} U_3, \cdot), \quad r = s_1 + s_2 + s_3 + 1.$$

**Lemma**

At $p$,

- $\nu_2, \nu_3$ agree with the flat model to order 3,
- $\nu_0, \nu_1$ agree with the flat model to order 4.
Image of singular locus

\[ G_2 \text{ case} \]

\[ d\nu_1 = \varphi(U_2, U_3, \cdot), \quad d\nu_2 = \varphi(U_3, U_1, \cdot) \]
\[ d\nu_3 = \varphi(U_1, U_2, \cdot), \quad d\nu_0 = \ast\varphi(U_1, U_2, U_3, \cdot) \]

If \( U_1 \) vanishes on a collection of singular orbits, then \( \nu_2, \nu_3 \) and \( \nu_0 \) are locally constant on that collection.

- \( T^2 \) stabiliser \( \mapsto \) a point in \( \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \)
- \( S^1 \) stabiliser \( \mapsto \) lines in \( (\nu_0 = \text{constant}) \) of rational slope
- Any intersection is triple, with the primitive slope vectors summing to zero

Thus we get a collection of trivalent graphs.
**Complete $G_2$ Examples**

**Example**

Flat model $S^1 \times \mathbb{C}^3$: planar

**Example**

Bryant-Salamon metrics on $S^3 \times \mathbb{R}^4$: non-planar
**Example**

Foscolo et al. (2018) examples on $M_{m,n}$ have $M_{m,n}$ a circle bundle over the canonical bundle of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with first Chern class $(m, -n)$ over the zero section, symmetry group $\text{SU}(2) \times \text{SU}(2) \times S^1$:

Primitive directions

$(m - n, 0, n)$

$(0, n - m, m)$

$(n - m, m - n, -m - n)$

planar
**Explicit metrics with special holonomy**

Full holonomy $G_2$

\[ g = \frac{1}{\nu_0} (\theta_1^2 + \theta_2^2 + \theta_3^2) + \nu_0^2 (d\nu_1^2 + d\nu_2^2 + d\nu_3^2) + \nu_0^3 d\nu_0^2 \]

\[ d\theta_i = d\nu_j \wedge d\nu_k, \quad (ijk) = (123) \]

Full holonomy $\text{Spin}(7)$

\[ g = \frac{1}{\nu_1} \theta_0^2 + \frac{1}{\nu_2} \theta_1^2 + \frac{1}{\nu_3} \theta_2^2 + \frac{1}{\nu_0} \theta_3^2 \]

\[ + \nu_2 \nu_3 \nu_0 d\nu_0^2 + \nu_1 \nu_3 \nu_0 d\nu_1^2 + \nu_1 \nu_2 \nu_0 d\nu_2^2 + \nu_1 \nu_2 \nu_3 d\nu_3^2 \]

\[ d\theta_0 = -\nu_2 d\nu_{23}, \quad d\theta_1 = -\nu_3 d\nu_{03}, \quad d\theta_2 = -\nu_0 d\nu_{01}, \quad d\theta_3 = \nu_1 d\nu_{12} \]
References I


