A presentation of the Gfan software package

Anders Nedergaard Jensen *

Department of Mathematical Sciences, University of Aarhus and
Institute for Operations Research, ETH Zürich

8th April 2005

Abstract

Gfan is a new software package for computing Gröbner fans of polynomial ideals in $\mathbb{Q}[x_1, \ldots, x_n]$. We give a short description of this package. Some technical details are given to give the reader an idea of what the software can do.

1 Background

Introduction Gfan is a software package for computing the *Gröbner fan* ([9]) of a given polynomial ideal. It is an implementation of the ideas appearing in [4] which is joint work with Komei Fukuda and Rekha Thomas. For toric and lattice ideals such programs already exist: TiGERS [6] and CaTS [7]. Gfan works on any ideal in $\mathbb{Q}[x_1, \ldots, x_n]$. Besides Buchberger's algorithm, the local basis change procedure [2] and the simplex method, the reverse search technique [1] and algorithms for exploiting symmetry are used. This allows enumeration of fans with millions of cones. Gfan has been used for studying the structure of the Gröbner fan. Among the new results is an example of a Gröbner fan which is not the normal fan of a polyhedron [8].

The Gröbner fan of an ideal The Gröbner fan of an ideal $I \subset k[x_1, \ldots, x_n]$ is a polyhedral complex consisting of cones in $\mathbb{R}^n$. The monomial initial ideals (with respect to term orders) of I are in bijection with the marked reduced Gröbner bases of I and with the full dimensional cones in the Gröbner fan of I. Knowing a marked reduced Gröbner basis its initial ideal and equations defining its Gröbner cone are easily read off. Thus a useful way to present the Gröbner fan of an ideal is by the set of its reduced Gröbner bases.

*Partially supported by the Faculty of Science, University of Aarhus, Danish Research Training Council (Forskeruddannelsesrådet, FUR) , Institute for Operations Research ETH, grants DMS 0222452 and DMS 0100141 of the U.S. National Science Foundation and the American Institute of Mathematics.

2 Demonstration of the software

Computing all reduced Gröbner bases Computing all reduced Gröbner bases of a polynomial ideal is the primary purpose of the software. This can be done using the program `gfan`. For example, running

```
gfan
```

on the input

```
{a^2+b*c, b^2+a*c, c^2+a*b}
```

produces a list of the 9 reduced Gröbner bases of the ideal generated by the input:

```
{{c^4, b*c^2, b^2*c, b^3-c^3, a*c+b^2, a*b+c^2, a^2+b*c},
{c^3-b^3, b*c^2, b^2*c, b^4, a*c+b^2, a*b+c^2, a^2+b*c},
{c^2+a*b, b^2*c, b^4, a*c+b^2, a*b^2, a^2+b*c},
{c^2+a*b, b*c+a^2, b^4, a*c+b^2, a*b^2, a^2*b, a^3-b^3},
{c^2+a*b, b*c+a^2, b^3-a^3, a*c+b^2, a*b^2, a^2*b, a^4},
{c^4, b*c^2, b^2+a*c, a*c^2, a*b+c^2, a^2+b*c},
{c^4, b*c+a^2, b^2+a*c, a*c^2, a*b+c^2, a^2*c, a^3-c^3},
{c^3-a^3, b*c+a^2, b^2+a*c, a*c^2, a*b+c^2, a^2*c, a^4},
{c^2+a*b, b*c+a^2, b^2+a*c, a^2*c, a^2*b, a^4}}
```

Combining programs on the command line Since Gfan is not part of a big algebra program we are limited to doing manipulations supported by the UNIX shell. The Gfan package contains other programs than `gfan`. For example we may combine `gfan` and `gfan_polynomialsetunion` using the pipe operation to compute a *universal* Gröbner basis:

```
gfan | gfan_polynomialsetunion
```

With the same input as before the output will be

```
{c^4, b*c^2, b^2*c, b^3-c^3, a*c+b^2, a*b+c^2,
a^2+b*c, c^3-b^3, b^4, a*b^2, a^2*b, a^3-b^3,
b^3-a^3, a^4, a*c^2, a^2*c, a^3-c^3, c^3-a^3}
```

Another possibility is to visualize the Gröbner fan.

```
gfan | gfan_render > picture1.fig
```

will produce the first xfig file shown in Figure 1 while the following will render the staircase diagrams:

```
gfan | gfan_renderstaircase -m > picture2.fig
```

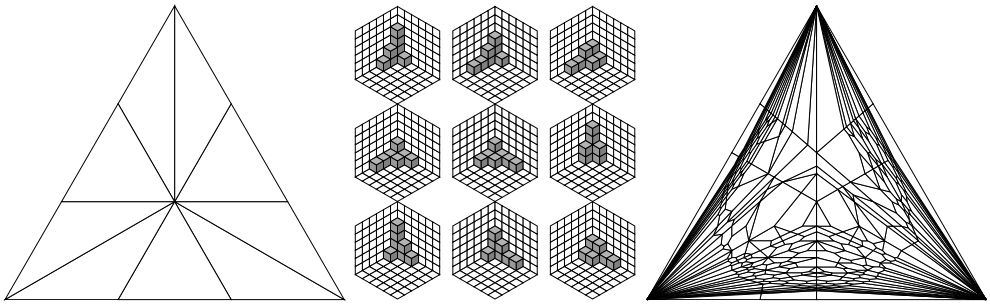

Figure 1: The Gröbner fan of the ideal intersected with the standard simplex, staircase diagrams visualising the various monomial initial ideals and a Gröbner fan of a different ideal.

3 Advanced features

Symmetry Our examples are rarely random. They often possess a lot of symmetry. In our example, the ideal is invariant under any permutation of the three variables. In many cases it would be sufficient to know only the essentially different Gröbner bases. Gfan can do its computations up to symmetry.

Interactive mode The program `gfan_interactive` is useful for investigating the local structure of the fan. It allows the user to interactively walk from cone to cone along an arbitrary path in the Gröbner fan.

4 Final remarks

Performance The software has been used to compute big examples. Here is an example of its performance: The ideal generated by the 4×4 minors of 4×5 matrix has 3000 reduced Gröbner bases. They can be computed in 5 hours using reverse search. Exploiting the symmetry of the ideal they can be computed in 12 seconds.

Future improvements A natural extension of the software is a program for checking if a set of polynomials is a Gröbner basis with respect to *some* (unknown) term order. Another could be a *universal* Gröbner basis test - is a given set of polynomials a Gröbner basis with respect to *any* term order?

Supported platforms and required libraries The following platforms are supported: Linux, Mac OS X and other UNIX variants. It is likely that the program will also run on Microsoft Windows systems through Cygwin. The program is written in C++ and can be compiled with gcc 3.3.3. The following libraries are required: gmp [5] (arithmetics) and cddlib [3] (LP-solving).

References

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