HyperKähler metrics and symmetries

Andrew Swann

Department of Mathematics & DIGIT, Aarhus University

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Section 1

HyperKähler geometry

HyperKähler manifolds

HyperKähler manifold: (M, g) Riemannian, I, J, K compatible almost complex structures parallel for the Levi-Civita connection and with $IJ = K = -JI$.

Consequently: (a) I, J, K are integrable, (b) $\omega_I = g(I\,\cdot, \cdot\,)$ etc. are symplectic forms.

$$\text{hyperKähler} \begin{cases} \subset \text{complex symplectic} & \subset \text{complex} \\ \quad \omega_{\mathbb{C}} = \omega_J + i\omega_K & \\ \subset \text{Calabi-Yau} & \subset \text{Ricci-flat} \\ \quad \Omega = \omega_{\mathbb{C}}^n & \\ \subset \text{tri-symplectic} & \subset \text{symplectic} \\ \subset \text{quaternionic Kähler} & \subset \text{Einstein} \end{cases}$$

Despite more rigidity, many constructions from symplectic geometry have hyperKähler analogues.

A central reason is that algebraically compatible $\omega_I, \omega_J, \omega_K$ with $d\omega_I = 0 = d\omega_J = d\omega_K$ gives hyperKähler (cf. Atiyah and Hitchin 1988)

Quaternions and actions

$\mathbb{H} = \mathbb{R}^4$ basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{i}^2 = -1 = \mathbf{j}^2 = \mathbf{k}^2$, quaternion division algebra.

Standard flat hyperKähler example: $\mathbb{H}^n$, $Iq = -q\mathbf{i}$, etc., and with standard inner product.

$\mathrm{Sp}(1) = \mathrm{SU}(2) = \{a\mathbf{i} + b\mathbf{j} + c\mathbf{k} : a^2 + b^2 + c^2 = 1\}$ acts on the right.
$\mathrm{Sp}(n) = \{A \in M_n(\mathbb{H}) \mid \overline{A}^T A = 1_n\}$ is centraliser in $\mathrm{SO}(4n)$ of $\mathrm{Sp}(1)$.
A hyperKähler manifold is a Riemannian $4n$-manifold with holonomy in $\mathrm{Sp}(n)$.

If X is Killing, then $\nabla X \in \mathfrak{so}(4n)$ normalises the holonomy algebra.
$\mathfrak{sp}(n)$ has normaliser $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$.
$(\nabla X)^{\mathfrak{sp}(1)}$ determines the action on $\mathrm{Span}\{\omega_I, \omega_J, \omega_K\}$.

Two types of Killing vector field to consider
triholomorphic $L_X\omega_I = 0 = L_X\omega_J = L_X\omega_K$, or
rotating $L_X\omega_I = 0$, $L_X\omega_J = \omega_K$, $L_X\omega_K = -\omega_J$.

Symmetries and quotients

For X Killing on Riemannian (M,g) $R_{A,X}B = \nabla^2_{A,B}X$, so $\mathrm{Ric} = 0$ implies $\Delta X = 0$. If M is compact, then X is parallel and M has a flat factor.

So will usually consider (M,g) non-compact. May impose a completeness condition.

For X triholomorphic, there is locally a *moment map* $\mu\colon U \longrightarrow \mathbb{R}^3$ with

$$d\mu = (d\mu_I, d\mu_J, d\mu_K) = (X \lrcorner\, \omega_I, X \lrcorner\, \omega_J, X \lrcorner\, \omega_K)$$

For μ defined globally, X is *tri-Hamiltonian*. G is tri-Hamiltonian if $\mu\colon M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$ is also equivariant.

Theorem (Hitchin et al. 1987)

For G tri-Hamiltonian, $M/\!/\!/G = \mu^{-1}(0)/G$ is hyperKähler.

Section 2

The Gibbons-Hawking construction

The Gibbons-Hawking Ansatz

Gibbons and Hawking (1978): Any hyperKähler metric g in dimension four with a tri-holomorphic Killing vector field X away from zeros has the form

$$g = \frac{1}{V}\theta^2 + V\,(dx^2 + dy^2 + dz^2)$$

$\mu = (x, y, z)$, etc., $V = 1/g(X, X)$, $\theta(X) = 1$.
The hyperKähler condition is

$$d\theta = -*_3 dV.$$

Implies V is a positive harmonic function on $U \subset \mathbb{R}^3$.

Theorem (Bielawski 1999; Swann 2016)

M^4 complete hyperKähler with tri-Hamiltonian circle action have $\mu\colon M^4/S^1 \to \mathbb{R}^3$ a homeomorphism and are given by

$$V(p) = c + \frac{1}{2}\sum_{q\in Q}\frac{1}{\lVert p-q\rVert}, \quad p \in \mathbb{R}^3$$

$c \geqslant 0$ (Taub-NUT parameter), $Q \subset \mathbb{R}^3$, $V(p) < +\infty$ for some $p \in \mathbb{R}^3$.

Examples

$Q = \{0\}$, $c = 0$: (M^4, g) is flat $\mathbb{R}^4 = \mathbb{H}$.
$Q = \emptyset$, $c > 0$: $M^4 = S^1 \times \mathbb{R}^3 = \mathbb{R}^4/(z \mapsto z + \mathbf{e}_1)$ flat
$Q = \{0\}$, $c > 0$: Taub-NUT metric on $\mathbb{R}^4$
$Q = \{\mathbf{e}_1, -\mathbf{e}_1\}$, $c = 0$: Eguchi-Hanson or Calabi metric on $T^*\mathbb{C}P(1)$.
$Q = \{n^2\mathbf{e}_1 \mid n \in \mathbb{Z}_{>0}\}$, $c = 0$: metric of Anderson et al. 1989 with $b_2(M) = \infty$
(see also Goto 1998; Hattori 2011)

Question

Local considerations give $\mu\colon M^4/S^1 \longrightarrow \mathbb{R}^3$ is a local homeomorphism. Global result above has two ingredients:

1. (Bielawski) μ gives a conformal immersion $\mu\colon M_0^4/S^1 \longrightarrow \mathbb{R}^3$; the metric on M_0^4/S^1 has non-negative scalar curvature and can be modified to also be complete; use a result of Schoen and Yau (1994) to get μ is injective and $\partial(\mu(M_0^4))$ has Newtonian capacity 0
2. using the Martin boundary representation of harmonic functions, get growth constraints on V, and then completeness of $Vg_{\mathbb{R}^3}$ to get $\partial(\mu(M^4))$ is empty.

Problem

Is there a direct proof that $\mu\colon M^4/S^1 \longrightarrow \mathbb{R}^3$ is injective? or surjective?

Section 3

Cotangent bundles

Cotangent bundles

For N a complex manifold, $T^*N = \Lambda^{1,0}N$ carries a complex symplectic structure $\omega_{\mathbb{C}}$. Does it carry a compatible hyperKähler structure?

There are at least three different situations where the is a positive answer.

The most general is

Theorem (Feix 2001; Kaledin 2001)

Suppose N is real analytic and Kähler, then a neighbourhood of the zero section in T^*N admits a hyperKähler metric compatible with $\omega_{\mathbb{C}}$.

Unique characterisation: restricts to the given Kähler structure on the zero section, and multiplication by S^1 in the fibres is a rotating isometry. In general, the metric is not complete, but can be controlled in some circumstances cf. Abasheva 2020.

Special Kähler

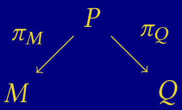
Other extreme, if (N, g, ω, I_N) is Hermitian with a connection ∇, then $TT^*N \cong TN \oplus T^*N$ carries $\omega_I := \omega + (\omega)^*$.

Theorem (Freed 1999,…)

$(\omega_I, \omega_{\mathbb{C}})$ defines a hyperKähler structure on T^*N if and only if (N, ω, ∇, I_N) is special Kähler: ∇ is flat, torsion-free, $\nabla\omega = 0$, $(\nabla_X I_N)Y = (\nabla_Y I_N)X$.

If the special Kähler structure on N is *conic*, meaning there is a Killing vector field with $\nabla X = -I_N = \nabla^{LC} X$, then this lifts horizontally to a rotating action of the hyperKähler structure on T^*N.

THE qK/hK CORRESPONDENCE



M hyperKähler with rotating X, Kähler moment map μ_I.
P an S^1-bundle over M with connection, principal action generated by Y, horizontal space $\mathcal{H}$, curvature form $F = dX^\flat + \omega_I$.

Then $Q = P/\langle X_{\mathcal{H}} + aY\rangle$, $a = \|X\|^2 - \mu_I + c$, is quaternionic Kähler with isometry $(\pi_Q)_* Y$, where $\pi_Q^* g_Q = \pi_M^* \widetilde{g}_M$ on $\mathcal{H}$ with $\widetilde{g}_M$ the *elementary deformation*

$$\widetilde{g}_M = \frac{1}{\mu_I - c} g_M - \frac{1}{(\mu_I - c)^2} \frac{g|_{\mathbb{H}X}}{\|X\|^2} \qquad \text{of } g_M$$

THEOREM (HAYDYS 2008; MACIÁ AND SWANN 2015)

Every quaternionic Kähler metric with an isometry is locally of this form and these are the only elementary deformations that produce quaternionic Kähler metrics this way.

Note to get $\widetilde{g}_M$, and hence g_Q, positive definite may need g_M to have indefinite signature.

The qK/hK correspondence, using cotangent bundles of conic special Kähler manifolds, produces all known left-invariant quaternionic Kähler structures (negative scalar curvature) on Lie groups (Cortés 1996; de Wit and Van Proeyen 1992).

Other complete quaternionic Kähler manifolds, e.g. of lower cohomogeneity, can be produced cf. Cortés et al. 2017

Problem

Use the qK/hK correspondence to classify homogeneous quaternionic Kähler manifolds.

Section 4

Coadjoint orbits

The Calabi metric

Calabi's example $T^*\mathbb{C}P(n)$.
May be obtained as hyperKähler quotient of S^1 acting on $\mathbb{H}^n$, $q \mapsto e^{\mathbf{i}\theta}q$.

$$\mu(z + \mathbf{j}w) = \left(\tfrac{1}{2}(|z|^2 - |w|^2), z^T w\right) + c.$$

$T^*\mathbb{C}P(n)$ is also the orbit of

$$\begin{pmatrix} \mathbf{i}1_n & 0 \\ 0 & -n\mathbf{i} \end{pmatrix} \in \mathfrak{sl}(n+1, \mathbb{C})$$

The Killing form of $\mathfrak{sl}(n+1, \mathbb{C})$ is non-degenerate, so this may also be identified with an orbit in $\mathfrak{sl}(n+1, \mathbb{C})^*$.

Coadjoint orbits

For $G_{\mathbb{C}}$ a Lie group over $\mathbb{C}$, each orbit $\mathcal{O} \subset \mathfrak{g}_{\mathbb{C}}^*$ admits a Kirillov-Kostant-Souriau complex symplectic form

$$\omega_{\mathbb{C}}(A_\varphi, B_\varphi) = \varphi([A, B]).$$

Theorem (Biquard 1996; Kovalev 1996)

For G compact, semi-simple, every $G_{\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$ carries a G-invariant hyperKähler metric.

Theorem (Kronheimer 1986)

For G compact, $T^*G_{\mathbb{C}}$ carries a complete G-invariant hyperKähler metric on a star-shaped neighbourhood of the zero section.

Both results construct the metric as moduli space of solutions of Nahm's equations $T_1, T_2, T_3 \colon$ interval $\to \mathfrak{g}$

$$\frac{dT_1}{dt} = [T_2, T_3], \qquad \text{and cyclically.}$$

Problem

Give more direct ("finite") descriptions of these metrics.

All orbits of $\mathrm{SL}(n, \mathbb{C})$ may be obtained as finite-dimensional hyperKähler quotients

$$\mathbb{H}^N = \mathbb{C} \leftrightarrows \mathbb{C}^2 \leftrightarrows \cdots \leftrightarrows \mathbb{C}^{n-1} \leftrightarrows \mathbb{C}^n$$
$$G = \mathrm{U}(1) \times \mathrm{U}(2) \times \cdots \times \mathrm{U}(n-1)$$

Note $Z(G) = T^{n-1}$.
There are similar constructions for classical groups, but the center is small and essentially only get nilpotent orbits.

Problem

For $G_{\mathbb{C}}$ non-reductive do there exist hyperKähler metrics on $\mathcal{O} \subset \mathfrak{g}_{\mathbb{C}}^*$ compatible with $\omega_{\mathbb{C}}$?

Allow metrics of different signature.
Allow degenerations (and signature changes) along hypersurfaces.

Existence of hyperKähler structure on $\mathcal{O}$ would enable some form of construction of hyperKähler quotients at non-zero levels $\lambda \in \mathfrak{g}^* \otimes \mathbb{R}^3$

$$M///_{\lambda}G \sim M \times \mathcal{O}_{-\lambda_{\mathbb{C}}}///G$$

Note there are now certain non-reductive versions of GIT that are relevant (Bérczi et al. 2018; Doran and Kirwan 2007).

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