Special Geometries and Moment Maps

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Holonomy Classification: every Riemannian manifold is locally a product of:

Hol_0	dim M	Name	Curvature	Degree of defining forms	
$SO(n)$	n	generic	·	n	
$U(m)$	$2m$	Kähler	·	(2)	(symplectic)
$SU(m)$	$2m$	Calabi-Yau	Einstein scal $=0$	$2, m$	
$Sp(r)$	$4r$	hyperKähler	Einstein scal $=0$	$2,2,2$	
$Sp(r)Sp(1)$	$4r > 4$	quaternionic Kähler	Einstein scal $\neq 0$	4	
G_2	7	exceptional	Einstein scal $=0$	3	
$Spin(7)$	8	exceptional	Einstein scal $=0$	4	
G^σ	$\dim G - \dim G^\sigma > 1$	Riemannian symmetric	Einstein scal $\neq 0$	—	

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1. Symplectic and HyperKähler Constructi

Symplectic Geometry

(M,ω) : $\omega \in \Omega^2(M)$ closed $d\omega = 0$
& non-degenerate $X \lrcorner \omega = \omega(X,\cdot) \neq 0 \ \forall X$

Darboux Theorem: locally (M^{2n},ω) is $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$
with $\omega = \sum_{j=1}^n dq^j \wedge dp_j$

(Co)adjoint orbits : G compact Lie, $M = \mathcal{O} = G\cdot X \subset$
is symplectic with Kirillov-Kostant-Souriau form
$\omega_{\mathcal{O}}([X,A],[X,B])_X = \langle X,[A,B]\rangle$

Moment maps G acting on M preserving ω
$X \in \mathfrak{g}$ induces X on M with
$0 = L_X\omega = X \lrcorner \underbrace{d\omega}_{0} + d(X\lrcorner\omega)$

$\mu : M \longrightarrow \mathfrak{g}^*$ is a moment map if
(1) $d\mu_X = d\langle\mu, X\rangle = X \lrcorner \omega$ $\forall X \in \mathfrak{g}$
& (2) μ is equivariant

G-action is then Hamiltonian

Examples : (a) $M = \mathbb{R}^2 = \mathbb{C}$ $\omega = i\, dz\wedge d\bar{z} = 2dx\wedge dy$
$G = S^1$, $z \to e^{ik\theta}z$ $\mu = ik|z|^2 + c$
(b) $(M = \mathcal{O}, \omega_{\mathcal{O}})$ $\mu : \mathcal{O} \hookrightarrow \mathfrak{g} \cong \mathfrak{g}^*$.

Symplectic reduction $M/\!/G = \mu^{-1}(0)/G$
is a stratified symplectic space (Sjamaar)

For $\lambda \in \mathfrak{g}$, $M/\!/_\lambda G := \mu^{-1}(\lambda)/\mathrm{Stab}_G(\lambda) = (M \times \mathcal{O}_{-\lambda})/\!/G$

HyperKähler Geometry

(M, g, I, J, K) with

(1) g Riemannian metric

(2) $I, J, K \in \mathrm{End}\, TM$ satisfying

(i) $I^2 = -1 = J^2 = K^2$, $IJ = K = -$

(ii) $g(A\cdot, A\cdot) = g(\cdot,\cdot)$ for $A = I, J,$

(3) $\omega_A(\cdot,\cdot) = g(A\cdot,\cdot) \in \Omega^2(M)$ is closed, $A = I, J, K$.

Remarks: (A) $\omega_I, \omega_J, \omega_K$ are symplectic forms, $\dim M = 4r$

(B) Hitchin: I, J, K are integrable & parallel

(C) $\omega_c = \omega_J + i\omega_K \in \Omega^{2,0}_I$ is a complex-symplectic form

(D) $\omega_c^r \in \Omega^{2r,0}_I$ is a parallel complex-volume form, so (M, g, I) is Calabi-Yau and Ricci-flat

(E) $\omega_I, \omega_J, \omega_K$ define

$$I = \omega_K^{-1} \circ \omega_J : TM \xrightarrow{\omega_J} T^*M \xrightarrow{\omega_K^{-1}} TM$$

and hence g

Example: $M = \mathbb{H}^n = \mathbb{R}^{4n} = \mathbb{C}^n + j\mathbb{C}^n$

$$\underline{\omega} = i\omega_I + j\omega_J + k\omega_K = d\bar{q}^T \wedge dq$$

$$q = z + jw \implies \omega_I = i(dz^T \wedge d\bar{z} - dw^T \wedge d\bar{w}$$

$$\omega_c = \omega_J + i\omega_K = 2\, dz^T \wedge dw$$

HyperKähler Moment Maps

$(M, \underline{\omega} = i\omega_I + j\omega_J + k\omega_K)$ hyperKähler with a tri-Hamiltonian action of G has hyperKähler moment map

$$\underline{\mu} = i\mu_I + j\mu_J + k\mu_K : M \longrightarrow \mathfrak{g} \otimes \operatorname{Im} \mathbb{H}$$
$$= \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$$

(1) $d\underline{\mu}_X = d\langle \underline{\mu}, X \rangle = X \lrcorner \underline{\omega} \qquad \forall X \in \mathfrak{g}$

(2) $\underline{\mu}$ is equivariant under G
and under $SO(3)$ acting on
$\operatorname{Im}\mathbb{H}$ & $\operatorname{Span}_{\mathbb{R}}\{I,J,K\} \cong \mathbb{R}^3$

Examples: (A) $M = \mathbb{H} \quad G = \mathbb{R} \qquad q \to q + t, \ t \in \mathbb{R}$

$$X = \frac{1}{2}\left(\frac{\partial}{\partial q} + \frac{\partial}{\partial \bar{q}}\right)$$

$$X \lrcorner \omega = X \lrcorner (d\bar{q} \wedge dq) = \frac{1}{2}(dq - d\bar{q})$$

$$\underline{\mu} = \operatorname{Im} q + c$$

$\operatorname{Im} \underline{\mu} = \operatorname{Im}\mathbb{H} = \mathbb{R}^3$

(B) $M = \mathbb{H} \quad G = S^1 \qquad q \mapsto e^{i\theta} q$

$$X = \frac{1}{2}\left(iq \cdot \frac{\partial}{\partial q} - \frac{\partial}{\partial \bar{q}} \cdot \bar{q} i\right)$$

$$X \lrcorner \underline{\omega} = -\frac{1}{2}\left(d\bar{q} \cdot iq - \bar{q} i \cdot dq\right)$$

$$\underline{\mu} = -\frac{1}{2}\bar{q} i q + c$$

$q = z + jw$

$$= -i\,\frac{1}{2}(|z|^2 - |w|^2) + j(\bar{i} z w)$$

$\operatorname{Im} \underline{\mu} = \operatorname{Im}\mathbb{H} = \mathbb{R}^3$

$\underline{\mu}^{-1}(\underline{\mu}(q)) = \{ e^{i\theta} \ldots$

HyperKähler Quotients & Modifications

$\underline{\mu}^{-1}(0)$ has normal bundle spanned by IX, JX, KX $\quad \forall X \in \mathfrak{g}$.

<u>Theorem:</u> $M /\!/\!/ G = \underline{\mu}^{-1}(0)/G$ is a hyperKähler space

i.e. a metric space that is a locally finite union of hyperKähler manifolds

Hitchin, Karlhede, Lindström, Roček ; Dancer, Sw

For G compact acting freely & M complete the quotient $M /\!/\!/ G$ is smooth & complete

<u>Example:</u> $M = (S^1 \times \mathbb{R}^3) \times \mathbb{H}$

$$S^1 \times \mathbb{R}^3 = \mathbb{H} /_{q \mapsto q + 2\pi}$$

S^1 acts by $([q], p) \longmapsto ([q+t], e^{it} p)$

$$[q] \in S^1 \times \mathbb{R}^3, \quad p \in \mathbb{H}$$

$$\underline{\mu}([q], p) = \operatorname{Im} q - \tfrac{1}{2} \bar{p} i p$$

$$\underline{\mu}^{-1}(0) = \left\{ ([q], p) : \operatorname{Im} q = \tfrac{1}{2} \bar{p} i p \right\}$$

$$\underline{\mu}^{-1}(0)/S^1 = \left\{ (([x], \tfrac{1}{2} \bar{p} i p), p) : \begin{array}{l} x \in \mathbb{R} \\ p \in \mathbb{H} \end{array} \right\} \Big/ \begin{pmatrix} x \mapsto x + t \\ p \mapsto e^{it} p \end{pmatrix}$$

$$\cong \mathbb{H} \qquad \qquad \downarrow \; p$$

Get a complete hyperKähler metric on $\mathbb{H} = \mathbb{R}^4$ that is not flat — the Taub-NUT metric.

<u>Example</u> : $M = \mathbb{H} \times \mathbb{H}$

$G = S^1$ acting by $\underline{q} \longmapsto e^{i\theta}\underline{q}$

$\underline{\mu}(\underline{q}) = -\frac{1}{2}\,\underline{\bar{q}}^T i\,\underline{q} + c$ $\qquad c \in \mathrm{Im}\,\mathbb{H}$

$SO(3)$-equivariance $\Rightarrow$ can take $c = i\lambda/2$, $\lambda \in \mathbb{R}$

$$\underline{\mu}(\underline{z} + j\underline{w}) = \frac{i}{2}\left(\lambda - \|z\|^2 + \|w\|^2\right) + j\,i\,\underline{z}^T\underline{w}$$

For $\lambda > 0$

$\underline{z} + j\underline{w} \in \mu^{-1}(0) \iff \|z\|^2 = \lambda + \|w\|^2$ & $z^T w = 0$

$\Rightarrow z \neq \underline{0}$, $[z] \in \mathbb{C}P^{n-1}$ well-defined

$w \in T^*_{[z]}\mathbb{C}P^{n-1}$

$\mathbb{H}^{\hat{n}} /\!/\!/_\lambda S^1 = T^*\mathbb{C}P^{n-1}$ $\qquad$ Calabi metric or Eguchi-Hanson

λ measures the size of the zero section

Dancer & Swann:

<u>Definition</u>: for M hyper-Kähler with tri-Hamiltonian S^1-action, the <u>hyper Kähler</u> <u>modification</u> is

$$M_{mod} = (M \times \mathbb{H}) /\!/\!/ S^1$$ if S^1 acts freely on $\underline{\mu}^{-1}(0)$

1, $\pi_1(M) = 0 \Longrightarrow \pi_1(M_{mod}) = 0$

& $b_2(M_{mod}) = b_2(M) + 1$

2, $M_{mod} \supset (M \times \{0\}) /\!/\!/ S^1 = M /\!/\!/ S^1 = \hat{X}$ $\qquad$ codim 4

3, $M^* \xleftarrow{S^1} \underline{\mu}^{-1}(0)^* \xrightarrow{S^1} M_{mod}^*$

$M^* = M \setminus \underline{\mu}_M^{-1}(0)$, $\qquad M_{mod}^* = M_{mod} \setminus \hat{X}$ etc.

4, M_{mod} again has an S^1-action:

$S^1 \times S^1$ acts on $M \times \mathbb{H}$

$\Longrightarrow \quad S^1 = \dfrac{S^1 \times S^1}{S^1}$ acts on $M \times \mathbb{H} /\!/\!/ S^1$

5, For any $c \in \operatorname{Im} \mathbb{H}$, $\underline{\mu} + c$ is again S^1-equivariant, so can adjust $\underline{\mu}$ to avoid problems

1, = "adding a brane" 3, ~ "T-dual"

4, & 5, rely on $G = S^1$ being Abelian

Hyper Kähler Implosion

with Andrew Dancer & Frances Kirwan

for SU(2)

$$\mathbb{H}^2 = \mathbb{C}^2 + j\,\mathbb{C}^2$$

has a tri-Hamiltonian action of $SU(2) \times U$

$SU(2)$: $\underline{z} \longmapsto A\underline{z}$ $\qquad \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z_1 + j w_1 \\ z_2 + j w_2 \end{pmatrix}$

$$= \begin{pmatrix} a z_1 + b z_2 + j(\bar{a} w_1 + \bar{b} w_2) \\ -\bar{b} z_1 + \bar{a} z_2 + j(-b w_1 + a w_2) \end{pmatrix}$$

$S^1 = U(1)$: $\underline{z} \longmapsto e^{i\theta} \underline{z}$ $\qquad \underline{z} + j\underline{w} \longmapsto e^{i\theta} \underline{z} + j e^{-i\theta} \underline{w}$

$\langle \underline{\mu}^{SU(2)}(\underline{z}), A \rangle = -\frac{1}{2} \bar{\underline{z}}^T A \underline{z}$ $\qquad \underline{\mu}^{S^1}(\underline{z}) = -\frac{1}{2} \bar{\underline{z}} i \underline{z} + \underline{c}$

HyperKähler manifolds can be reduced in stages

$$M /\!/\!/ H \times K = (M /\!/\!/ H) /\!/\!/ K$$

Thus if M is hyperKähler with tri-Hamiltonian $SU(2)$-action then

$$M_{hKimpl} := (M \times \mathbb{H}^2) /\!/\!/ SU(2)$$

is a hyperKähler space with tri-Hamiltonian S^1-action.

$$M_{hKimpl} /\!/\!/_{\underline{c}} S^1 = (M \times \mathbb{H}^2) /\!/\!/_{\underline{c}} S^1 \times SU(2)$$

$$= (M \times (\mathbb{H}^2 /\!/\!/_{\underline{c}} S^1)) /\!/\!/ SU(2$$

$$= (M \times \widetilde{M}(-\underline{c})) /\!/\!/ SU(2)$$

$$\widetilde{M}(-\underline{c}) = \begin{cases} T^*\mathbb{CP}(1) & \underline{c} \neq \underline{0} \\ \mathbb{H}/\{\pm 1\} & \underline{c} = \underline{0} \end{cases}$$

$$= SL(2,\mathbb{C}) \cdot \begin{pmatrix} c_2 + ic_3 & 0 \\ 0 & -(c_2+ic_3) \end{pmatrix}^{c_{\neq}}$$

provided $\underline{c} = \underline{0}$ or $c_2 + ic_3 \neq 0$.

Abelian reduction of $M_{hKimpl.}$ captures non-Abelian reduction of M at non-zero levels.

SPECIAL GEOMETRIES & MOMENT MAPS

2. The c-map and the twist construction

Ferrara & Sabharwal

$$\begin{tikzcd}
\overline{M} \arrow[rr, dashed, leftrightarrow, "c \,\&\, s\text{-maps}"'] & & Q \\
M \arrow[u] & & H \arrow[ll] \arrow[u, dashed]
\end{tikzcd}$$

$\overline{M}$: projective special Kähler, $\dim_{\mathbb{R}} = 2n$

Q: quaternionic Kähler, $s < 0$, $\dim_{\mathbb{R}} = 4n+4$

M: (conic) special Kähler $(2n, 2)$

H: hyperKähler $(4n, 4)$

joint work with Óscar Macia
building on Hitchin, Alekseevsky, Cortes, Haydys, ...

Example: M moduli space of (Y, Ω):
Y compact Kähler 3-fold $c_1(Y) = 0$
$\Omega \in \Omega^{3,0}(Y)$ holomorphic nowhere vanishing

$$\dim_{\mathbb{C}} M = h^{2,1} + 1$$

The rigid c-map

M smooth manifold $\Longrightarrow$ T^*M is symplectic

$\Theta \in \Omega^1(T^*M)$ $\qquad \Theta_\alpha(A) = \alpha(\pi_* A)$

$\omega = d\Theta$ $\qquad T^*M \xrightarrow{\pi} M$
$\qquad\qquad\qquad\; \overset{\Psi}{\alpha}$

Locally, $\Theta = p_i \, dq^i$, $\omega = dp_i \wedge dq^i$

M complex $\Longrightarrow$ T^*M complex-symplectic

Either use $T^*M \cong \Lambda^{1,0}M$

or $\qquad \omega_c = \omega_2 + i\omega_3$

$\omega_2 = d\Theta$ $\qquad$ as above

$\omega_3 = d\Theta^I$ $\qquad \Theta^I_\alpha(A) = \alpha(I\pi_* A)$

M special Kähler $\Longrightarrow$ $H = T^*M$ is hyperKähler

ω_2, ω_3 as above

Special Kähler (g, J, ω, ∇)

(g, J, ω) Kähler

$\nabla\omega = 0$ $\qquad d^\nabla J = 0$ $\qquad J \in \Omega^1(M, TM)$

∇ torsion-free & flat

$$T\,T^*M = \mathcal{V} + \mathcal{H} \cong T^* + T$$

$$\omega_1 = \omega^* + \pi^*\omega$$

Remarks : (1) M signature $(2p, 2q)$
$\Rightarrow H = T^*M$ signature $(4p, 4q)$

(2) $\mathbb{R}^{2p+2q}$ acts translations in the fibres (locally) and preserves the hyperKähler structure

∇ gives flat local coordinates
$\omega = \sum \omega_{ij}\, dx^i \wedge dx^j$
constant coefficients.

(3) $e^{i\theta}$ acts on the fibres of T^*M permuting ω_2 & ω_3

Conic structures

M Kähler $\quad X$ is conic if

$$L_X I = 0 \qquad L_X g = g$$
$$L_{IX} I = 0 \qquad L_{IX} g = 0$$

so $\quad L_{IX}\omega = 0 \qquad L_X \omega = \omega$

If ϕ is a moment map for IX
then $\bar{M} = \phi^{-1}(c)/\langle IX\rangle$ is Kähler
provided $g(X,X) \neq 0$ on $\phi^{-1}(c)$

For $g(X,X) < 0$, $\quad M$ signature $(2p, 2q)$
$\Rightarrow \bar{M}$ signature $(2p, 2q-2)$

For conic X, IX <u>is</u> Hamiltonian:

$$(d\,\|IX\|^2)(A) = A\,g(IX, IX)$$

$$= 2\,g(\nabla^{LC}_A IX, IX)$$

$$= -2\,g(\nabla^{LC}_{IX} IX, A) \quad \text{as } IX \text{ is Killi[ng]}$$

$$\Longrightarrow \quad d\,\|IX\|^2 = -\,IX \lrcorner\, d(IX)^\flat$$

But $\quad L_X \omega = \omega$

$$\Longrightarrow \quad \omega = X \lrcorner\, \cancelto{0}{d\omega} + d(X \lrcorner\, \omega)$$

$$= d(g(IX, \cdot)) = d(IX)^\flat$$

So

$$d\,\|IX\|^2 = -\,IX \lrcorner\, \omega$$

$$\phi = -\|IX\|^2 = -\|X\|^2$$

is a moment map

Conic special Kähler M

$\equiv$ special Kähler + conic X

such that IX preserves ∇

$\bar{M}$ is then projective special Kähler

$$M \xrightarrow[\mathbb{C}^*]{} \bar{M}$$

- IX then induces a triholomorphic isometry of T^*M.
- X acts by scaling on the fibres of T^*M & on the base

The fibre action agrees with the natural $\mathbb{C}^*$ action on the fibre

so combining the actions of IX and the fibrewise $e^{i\theta}$ we get an isometry Z that preserves I permutes J & K and is trivial on fibres.

M hyperKähler X is conic if

$$\mathcal{L}_X I = 0 = \mathcal{L}_X J = \mathcal{L}_X K \qquad \mathcal{L}_X g = g$$

$$\mathcal{L}_{IX} I = 0 \qquad \mathcal{L}_{IX} J = K \qquad \mathcal{L}_{IX} g = 0$$
$$\mathcal{L}_{JX} J = 0 \qquad \mathcal{L}_{JX} K = I \qquad \mathcal{L}_{JX} g = 0$$
$$\mathcal{L}_{KX} K = 0 \qquad \mathcal{L}_{KX} I = J \qquad \mathcal{L}_{KX} g = 0$$

X generates an action of $\mathbb{H}^* = \mathbb{R}_{>0} \times SU$

$\mathbb{R}_{>0}$ tri-holomorphic
+ homothetic

$SU(2)$ isometric, permuting I, J, K

Let $\phi = \|X\|^2$, then

$\phi^{-1}(c)$ is 3-Sasakian (correct choice of c)
Einstein

&

$Q = \phi^{-1}(c)/SU(2)$
is quaternionic Kähler
Einstein orbifold

i.e. $\exists$ local $\omega_I, \omega_J, \omega_K$ with

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2 \quad \text{parallel}$$

Remark: • $\dim Q \geq 12$, $qK \Leftrightarrow d\Omega = 0$

• $\dim Q = 4$, qK is self-dual + Einstein

<u>Examples</u>: (a) $M = \mathbb{H}^n \qquad X = r\frac{\partial}{\partial r}$

$$Q = \mathbb{HP}^{n-1}$$

quaternionic projective space

(b) $M \subset \mathbb{H}^{n,1} \qquad X = r\frac{\partial}{\partial r} \qquad \|X\|^2 < 0$ on M

$$Q = \mathbb{H}H^{n}$$

quaternionic hyperbolic space

If M is quaternionic Kähler $(4n, 0)$ with non-zero scalar curvature then the Swann bundle $\mathcal{U}(M)$ is a hyperKähler cone

$\mathrm{scal}_M > 0$: signature $(4n+4, 0)$

$\mathrm{scal}_M < 0$: signature $(4n, 4)$

$$\mathcal{U}(M) = \mathbb{R}_{>0} \times \{ (\omega_I, \omega_J, \omega_K) : \text{compatible triples} \}$$

Isometries of M lift to triholomorphic isometries of $\mathcal{U}(M)$

The Haydys flip

Q quaternionic Kähler
with S^1-symmetry

gives

$\mathcal{U}(Q)$ hyperKähler with
tri-Hamiltonian S^1

then

$$H = \mathcal{U}(Q) /\!/\!/ S^1$$

is hyperKähler

and inherits an $S^1 \subset SU(2) \subset \mathbb{H}^*$
isometric, preserving I
permuting J & K.

Reversing this, keeping track of
signatures, gives the c-map

$$\begin{tikzcd}
\bar{M} \quad \text{psK} & & Q & \\
& & & \mathcal{U}(Q) \arrow[ul] \arrow[dl, "///"] \\
M \arrow[uu, "\mathbb{C}^*"] & & H = T^*M \arrow[ll] \arrow[uu, dashed] &
\end{tikzcd}$$

csK

The twist construction

$$\begin{tikzcd}[column sep=small] & P \arrow[dl, "S^1"'] \arrow[dr, "S^1"] & \\ Z \circlearrowright H & & Q \end{tikzcd}$$

P a principal S^1-bundle connection θ

curvature $\pi^*F = d\theta$

$F \in \Omega^2_{\mathbb{Z}}(H)$

with $L_Z F = 0$

and $Z \lrcorner F = -da \qquad a \in C^\infty(H)$

Given Z, F can choose P, θ, a so that

$$Z' = \tilde{Z} + aU$$

generator of principal action

defines a circle action.

Put $Q = P/\langle Z' \rangle$, then U acts on Q.

Invariant k-forms $\beta \in \Omega^k(H)$, $\beta_Q \in \Omega^k(Q$ are <u>$\mathcal{H}$-related</u> $\beta \sim_{\mathcal{H}} \beta_Q$ if

$$\pi_H^* \beta|_{\mathcal{H}} = \pi_Q^* \beta_Q|_{\mathcal{H}}$$

$\mathcal{H} = \ker \theta$

Lemma: $\beta \sim_{\mathcal{X}} \beta_Q$

$$\Rightarrow \quad d\beta_Q \sim_{\mathcal{X}} d\beta - \frac{1}{a} F \wedge (Z \lrcorner \beta)$$

Theorem: (H, g) hyperKähler with Z generating an isometric S^1-action preserving I and permuting J & K. Let μ be a Kähler moment map for Z.

Twisting $\left(H, \frac{1}{\mu} g + \frac{1}{\mu^2}(\alpha^2 + (I\alpha)^2 + (J\alpha)^2 + (K\alpha)^2)\right)$

$\alpha = Z^b$

w.r.t. $\quad F = dZ^b + \omega_I \qquad a = \|Z\|^2 - \mu$

gives a quaternionic Kähler metric on Q.

Moreover, for $\dim H \geqslant 12$, these are the only such choices of F and

$$g_H = f_1 \cdot g + f_2 \cdot (\alpha)^2_{\mathbb{H}}$$

Remark: hyperKähler modification is a twist of

$$g = g_{hK} + \frac{1}{\|\mu\|}(\alpha)^2_{\mathbb{H}}$$

SPECIAL GEOMETRIES & MOMENT MAP

3, Strong geometries & multi-moment maps

with Thomas Bruun Madsen

arXiv: 1012.2048, 1012.0402

(M, c) is <u>strong</u> if $c \in \Omega^3(M)$ is closed, $dc = 0$

If in addition

$$X \lrcorner c = 0 \Rightarrow X = 0$$

then (M, c) is <u>2-plectic</u>

(Baez, Hoffnung, Rogers)

Examples: ① $M = \Lambda^2 T^* N$

$$\begin{array}{c} \Lambda^2 T^* N \\ \downarrow \pi \\ N \end{array}$$

Canonical 2-form $b \in \Omega^2(M)$

$$b_\alpha(V, W) = \alpha(\pi_* V, \pi_* W) \qquad c = db$$

Locally, $\alpha = \sum_{i<j} p_{ij}\, dq^i \wedge dq^j$

$$c = \sum_{i<j} dp_{ij} \wedge dq^i \wedge dq^j$$

is 2-plectic

(2) SKT manifolds
"strong Kähler with torsion"

(g, I, ω) Hermitian
with $\partial\bar{\partial}\omega = 0$ $\quad c = -Id\omega$

<u>Gauduchon</u>: M^4 compact complex surface $\Rightarrow$ each Hermitian conformal class contains a 'unique' SKT metric

(3) SHKT manifolds

$(g, I, J, K, \omega_I, \omega_J, \omega_K)$

$c = -Id\omega_I = -Jd\omega_J = -Kd\omega_K$
$\Rightarrow I, J, K$ integrable
with $dc = 0$

E.g. most G compact Lie
$\dim G = 4n$
with biinvariant metric
& Joyce hypercomplex structure

$G = S^1 \times SU(2)$, $SU(3)$, ...

(4) manifolds with holonomy G_2 later

(5) (g, I, ω) Hermitian, $c = d\omega$

Basic calculations

(M, c) strong, G a group of symmetri[es]

$$0 = \mathcal{L}_X c = X \lrcorner \cancel{dc}^{\,0} + d(X \lrcorner c) \quad \forall X \in$$

$\Longrightarrow \quad X \lrcorner c \in \Omega^2(M)$ closed.

but often not exact

For $Y \in \mathfrak{g}$ with $[X, Y] = 0$

have

$$0 = \mathcal{L}_Y (X \lrcorner c)$$
$$= Y \lrcorner \cancel{d(X \lrcorner c)}^{\,0} + d(Y \lrcorner X \lrcorner c)$$
$$= d((X \wedge Y) \lrcorner c)$$

If for example $b_1(M) = 0$ then

$$X \wedge Y \lrcorner c = d\,\nu_{X \wedge Y}$$

for some $\nu_{X \wedge Y} \in C^\infty(M)$

Remark:

$$\{ X \wedge Y \ : \ X, Y \in \mathfrak{g}, \ [X, Y] = 0 \}$$

is a singular variety

Multi-moment maps

For $\mathfrak{g}$ a Lie algebra, the Lie kernel is

$$\mathcal{P}_{\mathfrak{g}} = \ker\left([\,,\,] : \Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}\right)$$

- a $\mathfrak{g}$-module
- typical element $p = \sum_{i=1}^{k} X_i \wedge Y_i$
  with $\sum_{i=1}^{k} [X_i, Y_i] = 0$.

Definition: a multi-moment map for G a symmetry group of a strong geometry (M, c) is a map

$$\nu : M \longrightarrow \mathcal{P}_{\mathfrak{g}}^{*}$$

such that

$$(1)\quad d\langle \nu, p\rangle = p \lrcorner c \qquad \forall p \in \mathcal{P}_{\mathfrak{g}}$$

and (2) ν is G-equivariant.

Examples: $\mathcal{P}_{\mathfrak{su}(2)} = \{0\}$

$$\mathcal{P}_{\mathfrak{su}(3)} = \Lambda^2\mathfrak{su}(3) \ominus \mathfrak{su}(3)$$

dim $= \frac{1}{2}.8.7 - 8 = 20$ irreducible

Existence & Uniqueness

Geometric criteria

(M, c) strong, G a group of symmetries
Then ν exists if either

(1) $c = db$ with b G-invariant

(2) $b_1(M) = 0$ & G is compact

or (3) $b_1(M) = 0$, M is compact & orientable & G preserves a volume form

Algebraic criteria

$$0 \to \mathcal{P}_{\mathfrak{g}} \xrightarrow{\iota} \Lambda^2\mathfrak{g} \xrightarrow{[\,,\,]} \mathfrak{g}$$

has dual

$$\mathfrak{g}^* \xrightarrow{d} \Lambda^2\mathfrak{g}^* \longrightarrow \mathcal{P}_{\mathfrak{g}}^* \longrightarrow 0$$

so $\mathcal{P}_{\mathfrak{g}}^* \cong \Lambda^2\mathfrak{g}^* / d(\mathfrak{g}^*)$

and $d : \Lambda^2\mathfrak{g}^* \longrightarrow \Lambda^3\mathfrak{g}^*$

induces $d_{\mathcal{P}} : \mathcal{P}_{\mathfrak{g}}^* \longrightarrow \Lambda^3\mathfrak{g}^*$.

Definition: $\mathfrak{g}$ is <u>$(2,3)$-trivial</u> if
$$b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$$

Then $d_{\mathcal{P}}$ is an isomorphism.
$$\mathcal{P}_{\mathfrak{g}}^* \longrightarrow \Lambda^3\mathfrak{g}^* \cap \ker d$$

<u>Theorem:</u> Suppose G acts nearly effectively on M preserving c.

(a) If G is $(2,3)$-trivial ~~then~~ and connected ~~then~~ ν exists and is unique

(b) If just $b_2(\mathfrak{g}) = 0$ and G is connected ~~then~~ ν is unique if it exists.

<u>Note:</u> G simple $\Rightarrow$ $b_2(\mathfrak{g}) = 0$
$b_3(\mathfrak{g}) = 1$

Proof ingredient: define $\Psi : M \to \Lambda^3 \mathfrak{g}^* \cap \ker$ by $\langle \Psi, X \wedge Y \wedge Z \rangle = c(X, Y, Z)$

<u>Structure Theorem:</u> $\mathfrak{g}$ is $(2,3)$-trivial if and only if

- $\mathfrak{g}$ is solvable
- $\mathfrak{k} = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ has codimension 1

and • $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}}$
$= H^3(\mathfrak{k})^{\mathfrak{g}}$

Note: $\mathfrak{k}$ is nilpotent.

$\Rightarrow$ • many examples of $(2,3)$-trivial $\mathfrak{g}$
• KKS theory for homogeneous 2-plectic manifolds

Torsion-free G_2-manifolds

The flat model

$$M = \mathbb{R}^7 = \operatorname{Im}\mathbb{O}$$

$$g_0 = \sum_{i=1}^{7} e_i^2$$

$$\phi_0 = e_{123} + e_1(e_{45}+e_{67}) + e_2(e_{46}-e_{57}) - e_3(e_{47}+e_{56}) \in \Lambda^3(\mathbb{R}^7)^*$$

$$e_{123} := e_1 \wedge e_2 \wedge e_3$$

$$\phi_0(a,b,c) = g_0(a, \operatorname{Im}(bc)) \qquad a,b,c \in \operatorname{Im}\mathbb{O}$$

$$G_2 = \{ g \in GL(7,\mathbb{R}) : g^*\phi_0 = \phi_0 \}$$

compact, simple, simply-connected Lie group of dimension 14

ϕ_0 determines g_0 & $\mathrm{vol}_0 = e_{1234567}$ via

$$(X \lrcorner \phi_0) \wedge (Y \lrcorner \phi_0) \wedge \phi_0 = 6\, g_0(X,Y)\, \mathrm{vol}_0 \qquad \forall X, Y \in \mathbb{R}^7$$

Also get a 4-form

$$*\phi_0 = e_{4567} + e_{23}(e_{67}+e_{45}) + e_{13}(e_{57}-e_{46}) - e_{12}(e_{56}+e_{47})$$

Multi-moment maps in the flat model

① $G = \mathbb{R}^2$ acting by translations

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}$$

$$\mathcal{P}_{\mathbb{R}^2} = \Lambda^2 \mathbb{R}^2 \cong \mathbb{R}$$

$$\nu : \mathbb{R}^7 \longrightarrow \mathcal{P}^*_{\mathbb{R}^2} = \mathbb{R}$$

$$d\nu = \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) \lrcorner \phi_0 = e_3 = dx^3$$

$$\nu = x^3 + c \qquad \nu^{-1}(0)/\mathbb{R}^2 = \mathbb{R}^4$$
$$= \mathrm{Span}\{e_4, e_5, e_6, \ldots$$

with
$$\omega_1 = \frac{\partial}{\partial x^1} \lrcorner \phi_0 = e_{45} + e_{67}$$
$$\omega_2 = \frac{\partial}{\partial x^2} \lrcorner \phi_0 = e_{46} - e_{57}$$
$$\omega_0 = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \lrcorner * \phi_0 = - e_{56} - e_{47}$$

self-dual 2-forms defining the same orientation.

② $G = T^2$ acting by rotations

$$T^2 \subset SU(3) \subset G_2$$

$$\mathbb{R}^7 = \mathbb{R} \overset{\perp}{\oplus} \mathbb{C}^3$$

$$\phi_0 = \frac{i}{2} dx \wedge (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$$
$$+ \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3)$$

$$U = \mathrm{Re}\left\{ i\left(z_1 \frac{\partial}{\partial z_1} - z_3 \frac{\partial}{\partial z_3}\right)\right\} \quad V = \mathrm{Re}\left\{ i\left(z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} \ldots\right.\right.$$

$$\nu(x, z_1, z_2, z_3) = -\frac{1}{4} \mathrm{Re}(z_1 z_2 z_3)$$

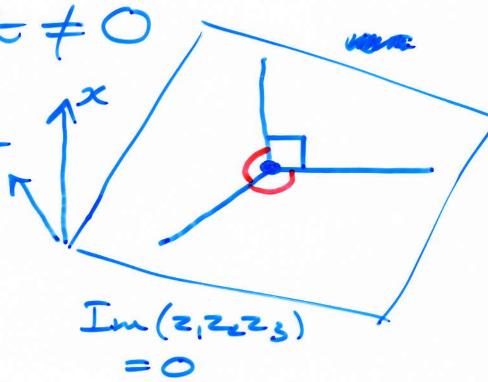
$$g_{uu} = \frac{1}{4}\left(\|z_1\|^2 + \|z_3\|^2\right) \qquad g_{vv} = \frac{1}{4}\left(\|z_2\|^2 + \|z_3\|^2\right)$$

$$g_{uv} = \frac{1}{4}\|z_3\|^2$$

$$\nu(x, z_1, z_2, z_3) = -\frac{1}{4}\mathrm{Re}(z_1 z_2 z_3)$$

$\nu^{-1}(t)/T^2$ is diffeomorphic to $\mathbb{R}^4$ for $t \neq 0$

$\nu^{-1}(0)/T^2$ is singular



The curved case

A G_2-structure on M^7 is a three-form $\phi \in \Omega^3(M)$ that is linearly equivalent on each T_xM to ϕ_0.

ϕ determines g, vol & $*\phi$

The G_2-structure is torsion-free if

$$d\phi = 0 \quad \& \quad d(*\phi) = 0$$

g is then Ricci-flat.

Toric G_2-manifolds

(M^7, ϕ) torsion-free G_2 with an effective T^2-symmetry, generators U, V, and a multi-moment map

$$\nu : M \longrightarrow \mathbb{R}$$

$$d\nu = U \wedge V \lrcorner \phi .$$

Assume the T^2-action is free.

Write $\quad g_{UV} = g(U,V) \quad$ etc.

Put $\quad h = \dfrac{1}{\sqrt{g_{UU}g_{VV} - g_{UV}^2}} > 0$

Define $\quad \omega_0 = U \wedge V \lrcorner *\phi$

$$\omega_1 = U \lrcorner \phi, \quad \omega_2 = V \lrcorner \phi$$

Theorem: Let $N^4 = \nu^{-1}(t)/T^2$ be a T^2-reduction of (M, ϕ). Then $\omega_0, \omega_1, \omega_2$ descend to symplectic 2-forms $\sigma_0, \sigma_1, \sigma_2$ defining the same orientation on N

$$h^2 \sigma_0^2 = \frac{1}{g_{UU}} \sigma_1^2 = \frac{1}{g_{VV}} \sigma_2^2 = \mathrm{vol}_N$$

$$\sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2 , \quad \sigma_1 \wedge \sigma_2 = 2 g_{UV} \mathrm{vol}_N$$

A coherent symplectic triple on a four-manifold N consists of 3 symplecti[c] forms $\sigma_0, \sigma_1, \sigma_2$ such that

$$\sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2$$

and $\sigma_0, \sigma_1, \sigma_2$ are pointwise linearly independen[t]

$$\begin{array}{ccc} T^2 \longrightarrow \pi^{-1}(t) & & \text{connection 1-forms } \theta_1, \theta_2 \\ \downarrow & & \text{satisfy} \\ N & & \begin{aligned} d\theta_1^+ &= a\sigma_1 + b\sigma_2 \\ d\theta_2^+ &= c\sigma_1 + d\sigma_2 \end{aligned} \end{array}$$

with

$$a q_{11} + b q_{21} + c q_{12} + d \cdot q_{22} = 0 \qquad (*)$$

where

$$\sigma_i \wedge \sigma_j = q_{ij}\, \sigma_0^2$$

Theorem: given a coherent symplectic trip[le] and curvature 2-forms $d\theta_1, d\theta_2$ satisfying $(*)$, the T^2-bundle X over N has a half-flat $SU(3)$-structure $\sigma \in \Omega^2(X), \psi_\pm \in \Omega^3($ given by

$d\psi_+ = 0$, $\sigma \wedge d\sigma = 0$

$$\sigma = h\sigma_0 + h^{-1}\theta_1 \wedge \theta_2$$

$$\psi_+ = \sigma_1 \wedge \theta_1 + \sigma_2 \wedge \theta_2$$

$$\psi_- = h^{-1}(q_{22}\sigma_1 \wedge \theta_2 - q_{11}\sigma_2 \wedge \theta_1 + q_{12}\sigma_1 \wedge \theta_1 - q_{12}\sigma_2 \wedge \theta$$

where $h = \sqrt{\det\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}} > 0$

Flowing this via the evolution equations

$$\psi_+' = d(h\sigma)$$
$$\left(\tfrac{1}{2}\sigma^2\right)' = -d(h\psi_-)$$

yields a torsion-free G_2-structure on $X \times (-\varepsilon, \varepsilon)$, when solutions exist, with T^2-symmetry and ν = projection to $(-\varepsilon, \varepsilon)$.

<u>Example</u> $(N, \sigma_0, \sigma_c = \sigma_1 + i\sigma_2, g_N)$ a complex symplectic Kähler surface

$$\Rightarrow \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$$

If $\sigma_1, \sigma_2 \in \Omega^2_{\mathbb{Z}}(N)$, then take

$$d\theta_1 = \sigma_2 , \quad d\theta_2 = -\sigma_1$$

Can solve the flow explicitly:

$$\phi = e^{2t}\sigma_0 \wedge dt + \theta_1(t) \wedge \theta_2(t) \wedge dt + e^t\left(\sigma_1 \wedge \theta_1(t) + \sigma_2 \wedge \theta_2(t)\right)$$

$$g = e^{4t}dt^2 + he^{2t}g_N + h^{-1}e^{-2t}(\theta_1^2 + \theta_2^2)$$

without explicit knowledge of a hyperKähler metric on N