Multi-moment maps for special Ricci-flat metrics

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Outline

Ricci-flat special holonomy

The Berger holonomy classification 1955,…, has only the following non-trivial irreducible Ricci-flat geometries

Name	Group	Dimension	Form degrees
Calabi-Yau	$SU(n)$	$2n$	$2, n, n$
HyperKähler	$Sp(n)$	$4n$	$2, 2, 2$
G_2 holonomy	G_2	7	3, 4
Spin(7) holonomy	Spin(7)	8	4

In the presence of symmetries, moment map techniques from symplectic geometry may be used if there is a closed form of degree 2, yielding many examples.

Symplectic constructions

Toric Calabi-Yau

Include symplectic quotients of $\mathbb{C}^N$ by subtori of T^N whose weights sum to zero.

$$\mathbb{C}^4 /\!/ \operatorname{diag}(e^{i\theta}, e^{i\theta}, e^{-i\theta}, e^{-i\theta}) = \big(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P(1)\big)$$

Hypertoric manifolds

Include hyperKähler quotients of $\mathbb{H}^N$ by subtori of T^N.

$$T^*\mathbb{C}P(n) = \mathbb{H}^{n+1} /\!/\!/ e^{i\theta} \mathbb{1}_{n+1}$$

Other constructions

On the other hand there are complete special holonomy metrics not obtained in such a way. These include the first complete examples found by Bryant and Salamon (1989)

M^7	$\Lambda^2_-(S^4)$	$\Lambda^2_-(\mathbb{C}P^2)$	$S^3 \times \mathbb{R}^4$
Isom_0	$\mathrm{Sp}(2)$	$\mathrm{SU}(3)$	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$
rank(Isom)	2	2	3

for G_2, and the Spin-bundle of S^4, $\mathrm{Isom}_0 = \mathrm{SO}(5) \times \mathrm{U}(1)$ of rank 3, for Spin(7).

Aim

Exploit forms of higher degree in such cases

Note: on compact manifolds, Ricci-flat implies that Killing vector fields are parallel and so the holonomy reduces. We will thus be interested in the non-compact situation.

Multi-Hamiltonian torus actions

(M, α) a manifold with a closed $\alpha \in \Omega^p(M)$ preserved by $G = T^n$ is *multi-Hamiltonian* if it there is a G-invariant

$$\nu \colon M \to \Lambda^{p-1}\mathfrak{g}^* \cong \mathbb{R}^N,$$
$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1} \rangle = \alpha(X_1, \ldots, X_{p-1}, \cdot\,)$$

for all $X_i \in \mathfrak{g}$.

- For $p = 2$ this is an ordinary symplectic moment map.
- ν invariant $\iff$ α pulls-back to 0 on each T^n-orbit
- $b_1(M) = 0 \implies$ each T^n-action preserving α is multi-Hamiltonian

More generally, we can consider several closed invariant forms $\alpha_k \in \Omega^{p_k}(M)$ with multi-moment maps ν_k and consider their product

$$\nu = (\nu_1, \ldots, \nu_m) \colon M \to \bigoplus_{k=1}^{m} \Lambda^{p_k - 1}\mathfrak{g}^*$$

An interesting case is when

$$\nu\colon M \longrightarrow \mathbb{R}^N$$

is of full rank on the part M_0 of M where $G = T^n$ acts freely, and

$$N = \dim(M_0/G).$$

Then ν locally exhibits M_0 as a principal T^n-bundle over $U \subset \mathbb{R}^N$.

Geometry	$\dim M$	$\deg \alpha$	G
Symplectic/Kähler	$2n$	2	T^n
Calabi-Yau	$2n$	$(2, n, n)$	T^{n-1}
HyperKähler	$4n$	$(2, 2, 2)$	T^n
G_2	7	$(3, 4)$	T^3
Spin(7)	8	4	T^4

Flat models

The flat symplectic/Kähler model is

- $M = \mathbb{C}^n$
- $\alpha = \omega = \sum_{k=1}^{n} dx_k \wedge dy_k = \frac{\mathbf{i}}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k = \frac{\mathbf{i}}{2} \sum_{k=1}^{n} dz_{k\overline{k}}$
- $G = T^n = \{\mathrm{diag}(e^{\mathbf{i}\theta_1}, \dots, e^{\mathbf{i}\theta_n})\}$
- $\nu = \mu = (\mu_1, \dots, \mu_n)$

$$\mu_k = \frac{1}{2}|z_k|^2$$

We have

$$\mu(\mathbb{C}^n) = [0, \infty)^n$$

and μ induces a homeomorphism

$$\mathbb{C}^n/T^n \longrightarrow [0, \infty)^n$$

But the latter is a manifold with corners.

Flat models, continued

For G_2 the flat model is

- $M = S^1 \times \mathbb{C}^3$
- $\alpha = (\varphi, *\varphi)$

$$\varphi = \frac{\mathbf{i}}{2} dx(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \mathrm{Re}(dz_{123})$$
$$*\varphi = \mathrm{Im}(dz_{123})dx - \frac{1}{8}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}})^2$$

- $G = T^3 = S^1 \times T^2 = S^1 \times \left\{ \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}, e^{\mathbf{i}\theta_3}) \,\middle|\, \theta_1 + \theta_2 + \theta_3 = 0 \right\}$

For Calabi-Yau the flat model is

- $M = \mathbb{C}^n$
- $\alpha = (\omega, \mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega), \quad \omega = \dfrac{\mathbf{i}}{2} \sum_{k=1}^{n} dz_{k\bar{k}}, \quad \Omega = dz_{12\ldots n}$
- $G = T^{n-1}$ = diagonal unitary matrices of determinant 1

Orbit spaces

For the G_2 case

$$M/G = (S^1 \times \mathbb{C}^3)/(S^1 \times T^2) = \mathbb{C}^3/T^2 = \operatorname{cone}(S^5)/T^2 = \operatorname{cone}(S^5/T^2)$$

And for the Calabi-Yau case

$$M/G = \mathbb{C}^n/T^{n-1} = \operatorname{cone}(S^{2n-1}/T^{n-1})$$

$$S^{2n-1} = \left\{ (r_1 e^{\mathbf{i}t_1}, \dots, r_{n-1} e^{\mathbf{i}t_{n-1}}) \;\middle|\; r_k \geqslant 0\,\forall k,\ \sum_{k=1}^{n-1} r_k^2 = 1 \right\}$$

Each T^{n-1}-orbit contains an element with $t_1 = t_2 = \cdots = t_{n-1}$ and that element is unique modulo $2\pi/n$ unless some r_k is zero.

Thus S^{2n-1}/T^{n-1} projects on to

$$\left\{(r_1^2,\dots,r_{n-1}^2)\,\middle|\, r_k \geqslant 0,\ \sum_{k=1}^{n-1} r_k^2 = 1\right\} = \Delta^{n-1} \equiv B^{n-1}$$

with fibres cirles over the interior, and points over the boundary. It follows that S^{2n-1}/T^{n-1} is homeomorphic to

$$\left\{(z,x)\in\mathbb{C}\times\mathbb{R}^{n-1}\,\middle|\,|z|^2+\|x\|^2=1\right\} = S^n$$

and $M/G = \mathbb{C}^n/T^{n-1}$ is homeomorphic to

$$\operatorname{cone}(S^n) = \mathbb{R}^{n+1}$$

Theorem

For all the multi-Hamiltonian geometries considered the torus actions has the property that every stabiliser is a connected subtorus.
Local models around any special orbit with stabiliser T^k are given by $(T^k \times \mathbb{R}^k) \times V$ where V is a flat model.

For example, in the Calabi-Yau case suppose $\dim \mathrm{Stab}_{T^{n-1}}(p) = k$. Then there are $n-1-k$ directions $U_1, \dots, U_{n-1-k}$ tangent to the orbit through p. But ω pulls-back to 0 on the orbit, so the U_i are linearly independent over $\mathbb{C}$. Now $\mathrm{Stab}_{T^{n-1}}(p)$ is an Abelian group acting on $T_pM = \mathbb{C}^n$ as a subgroup of $SU(n)$ and fixing a $\mathbb{C}^{n-1-k}$ pointwise, so a subgroup of $SU(k+1)$. But this forces it to be a maximal torus.

Corollary

For the Calabi-Yau, hyperKähler, G_2 and $\mathrm{Spin}(7)$ cases, M/G is homeomorphic to a smooth manifold.

via $\exp_p \colon T_pM \longrightarrow M$

Multi-moment maps for G_2

For G_2 the flat model is

- $M = S^1 \times \mathbb{C}^3$, $\alpha = (\varphi, *\varphi)$

$$\varphi = \frac{\mathbf{i}}{2} dx(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \mathrm{Re}(dz_{123})$$
$$*\varphi = \mathrm{Im}(dz_{123})dx - \frac{1}{8}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}})^2$$

- $G = T^3 = S^1 \times T^2$ generators

$$U_1 = \frac{\partial}{\partial x}, \qquad U_k = 2\,\mathrm{Re}\Big(\mathbf{i}\Big(z_k \frac{\partial}{\partial z_k} - z_3 \frac{\partial}{\partial z_3}\Big)\Big), \quad k = 2, 3$$

- $\nu = (\nu_1, \nu_2, \nu_3, \nu_0)$

$$d\nu_i = \varphi(U_j, U_k, \cdot\,) \quad (ijk) = (123), \qquad d\nu_0 = *\varphi(U_1, U_2, U_3, \cdot\,)$$
$$\nu_0 + \mathbf{i}\nu_1 = -\mathbf{i}z_1 z_2 z_3, \quad 2\nu_2 = |z_2|^2 - |z_3|^2, \quad 2\nu_3 = |z_3|^2 - |z_1|^2$$

Behaviour of flat model

Proposition

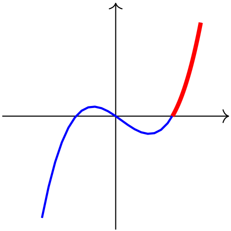
In the G_2 flat model, $\nu\colon M = S^1 \times \mathbb{C}^3 \to \mathbb{R}^4$

$$\nu_0 + \mathbf{i}\nu_1 = -\mathbf{i}z_1z_2z_3, \quad 2\nu_2 = |z_2|^2 - |z_3|^2, \quad 2\nu_3 = |z_3|^2 - |z_1|^2$$

induces a homeomorphism $M/G = \mathbb{C}^3/T^2 \to \mathbb{R}^4$.

This also applies to the Spin(7)-case. Similar results hold in the hyperKähler and Calabi-Yau cases.

Main point: for $t = |z_3|^2$, $c = \nu_0^2 + \nu_1^2$, satisfies $f(t) := t(t - 2\nu_3)(t + 2\nu_2) = c$ with each factor $\geqslant 0$. $(t, \nu) \mapsto \nu$ is a continuous bijection $\mathbb{R}^4 = \mathbb{C}^3/T^2 \to \mathbb{R}^5 \to \mathbb{R}^4$, so a homeomorphism, by Brouwer's invariance of domain.



General quotients via multi-moment maps

Theorem

For multi-Hamiltonian G_2, Spin(7) *and hyperKähler cases the multi-moment map* ν *induces local homeomorphisms*

$$M/G \longrightarrow \mathbb{R}^N$$

Also know it holds for Calabi-Yau cases when $n \leqslant 3$.
Ingredients in proof

- properties of commuting Killing vectors at zeros
- high-order approximation by the flat model
- local understanding of image sets of singular locus
- local injectivity argument at a point
- topological degree argument combined with deformation to flat model

Commuting Killing vector fields

X Killing implies

- ∇X is a skew-symmetric endomorphism of TM
- $\nabla^2_{A,B} X = -R_{X,A} B$

So $X_p = 0$ implies $(\nabla^2 X)_p = 0$ and $(\nabla^3 X)_p = -(R \circ \nabla X)_p$.
If X, Y are Killing, commute and $X_p = 0$, then

- ∇X and ∇Y commute at p.

G_2 case, with $\mathrm{Stab}_{T^3}(p) = T^2$, $T_pM = \mathbb{R} \oplus \mathbb{C}^3$. Can choose our generators so that U_2, U_3 are zero at p with covariant derivatives

$$(\nabla U_2)_p = \mathrm{diag}(\mathbf{i}, 0, -\mathbf{i}), \quad (\nabla U_3)_p = \mathrm{diag}(0, \mathbf{i}, -\mathbf{i}).$$

Let U be any generator that is non-zero at p. Then $\nabla U \in \mathfrak{g}_2$ and ∇U commutes with ∇U_i, $i = 1, 2$. But $\operatorname{rank} \mathfrak{g}_2 = 2$, so can adjust U to get at p U_1 unit length in $\mathbb{R}$ and $\nabla U_1 = 0$.

High-order approximation

G_2 case, $\mathrm{Stab}_{T^3}(p) = T^2$. At p, can ensure φ and $*\varphi$ agree with the flat model,

$$U_2 = 0 = U_3, \qquad \nabla U_1 = 0, \qquad \nabla^2 U_2 = 0 = \nabla^2 U_3$$

and $U_1, \nabla U_2, \nabla U_3$ agree with the flat model.
Now $dv_i = \varphi(U_j, U_k, \cdot\,)$, $(ijk) = (123)$, and $dv_0 = *\varphi(U_1, U_2, U_3, \cdot\,)$. But $\nabla\varphi = 0 = \nabla *\varphi$, so

$$\begin{aligned} \nabla^r v_i &= \varphi(\nabla^{s_1} U_j, \nabla^{s_2} U_k, \cdot\,), & r &= s_1 + s_2 + 1,\ (ijk) = (123) \\ \nabla^r v_0 &= *\varphi(\nabla^{s_1} U_1, \nabla^{s_2} U_2, \nabla^{s_3} U_3, \cdot\,), & r &= s_1 + s_2 + s_3 + 1. \end{aligned}$$

Lemma

At p,

- v_2, v_3 *agree with the flat model to order* 3,
- v_0, v_1 *agree with the flat model to order* 4.

Image of singular locus

G_2 case

$$dv_1 = \varphi(U_2, U_3, \cdot\,), \quad dv_2 = \varphi(U_3, U_1, \cdot\,)$$
$$dv_3 = \varphi(U_1, U_2, \cdot\,), \quad dv_0 = *\varphi(U_1, U_2, U_3, \cdot\,)$$

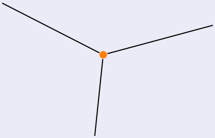
If U_1 vanishes on a collection of singular orbits, then v_2, v_3 and v_0 are locally constant on that collection.

- T^2 stabiliser $\mapsto$ a point in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$
- S^1 stabiliser $\mapsto$ lines in (v_0 = constant) of rational slope
- Any intersection is triple, with with the primitive slope vectors summing to zero

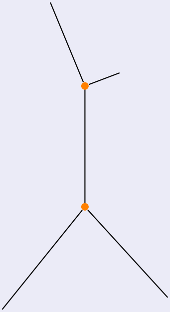
Thus we get a collection of trivalent graphs.

Complete G_2 examples

Example

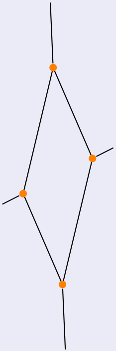
Flat model $S^1 \times \mathbb{C}^3$:  planar

Example

Bryant-Salamon metrics on $S^3 \times \mathbb{R}^4$:  non-planar

Example

Foscolo et al. (2018) examples on $M_{m,n}$ have $M_{m,n}$ a circle bundle over the canonical bundle of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with first Chern class $(m, -n)$ over the zero section, symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times S^1$:



Primitive directions
$(m - n, 0, n)$
$(0, n - m, m)$
$(n - m, m - n, -m - n)$
planar

Explicit metrics with special holonomy

Full holonomy G_2

$$g = \frac{1}{v_0}(\theta_1^2 + \theta_2^2 + \theta_3^2) + v_0^2(dv_1^2 + dv_2^2 + dv_3^2) + v_0^3 dv_0^2$$

$$d\theta_i = dv_j \wedge dv_k, \quad (ijk) = (123)$$

Full holonomy Spin(7)

$$g = \frac{1}{v_1}\theta_0^2 + \frac{1}{v_2}\theta_1^2 + \frac{1}{v_3}\theta_2^2 + \frac{1}{v_0}\theta_3^2 + v_2 v_3 v_0 dv_0^2 + v_1 v_3 v_0 dv_1^2 + v_1 v_2 v_0 dv_2^2 + v_1 v_2 v_3 dv_3^2$$

$$d\theta_0 = -v_2 dv_{23}, \quad d\theta_1 = -v_3 dv_{03}, \quad d\theta_2 = -v_0 dv_{01}, \quad d\theta_3 = v_1 dv_{12}$$

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