What is a Multi-moment Map?

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Joint work with Thomas Bruun Madsen

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1. Background
   - Symplectic Geometry
   - Strong Geometry
   - Covariant Moment Maps

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The moment map $\mu\colon \mathcal{O} \to \mathfrak{g}^*$ is just inclusion.

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Outline

1. Background
   - Symplectic Geometry
   - Strong Geometry
   - Covariant Moment Maps
2. Multi-moment maps
   - Commuting vector fields
   - Lie kernels
   - Existence
   - $(2,3)$-trivial Lie algebras
3. G_2 holonomy
   - Reduction
   - Conformal geometry

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Linearity in the basic calculation shows that

$$d(p \lrcorner c) = d(\sum_{i=1}^{r} c(X_i, Y_i, \cdot)) = -(\sum_{i=1}^{r} [X_i, Y_i]) \lrcorner c = 0.$$

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- For G Abelian, $\mathcal{P}_{\mathfrak{g}} = \Lambda^2 \mathfrak{g}$.
- For G semi-simple, $\Lambda^2 \mathfrak{g} \cong \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}$. In particular, for G compact and simple, $\mathcal{P}_{\mathfrak{g}}$ is the isotropy representation of the isotropy irreducible space $SO(\dim \mathfrak{g})/G$.

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$M = \Lambda^2 T^* N$, $G \subset \mathrm{Diff}(N)$.

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EXAMPLES II

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is the description of $SU(3)$ as a hypercomplex (HKT) Swann bundle over the quaternionic Kähler $\mathbb{C}P(2)$.

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Homogeneous spaces and orbits

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Outline

1. Background
   - Symplectic Geometry
   - Strong Geometry
   - Covariant Moment Maps
2. Multi-moment maps
   - Commuting vector fields
   - Lie kernels
   - Existence
   - $(2,3)$-trivial Lie algebras
3. G_2 holonomy
   - Reduction
   - Conformal geometry

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- There exist unimodular $(2,3)$-trivial Lie groups admitting compact discrete quotients ($\dim G \geqslant 5$).

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with

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One has

$$\frac{1}{\ell^2}\omega_0^2 = \frac{1}{\|U_i\|^2}\omega_i^2 = 2\,\mathrm{vol}_M,$$

$$\omega_0 \wedge \omega_i = 0, \quad \omega_1 \wedge \omega_2 = 2\langle U_1, U_2\rangle\,\mathrm{vol}_M\,.$$

Outline

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Let $\mathcal{X} \to M$ be a T^2-bundle with connection one-forms θ_i, $i = 1, 2$. Suppose $\lambda \in C^\infty(M)$ is positive. Then

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More general than Apostolov and Salamon (2004): we do not need a hyperKähler triple ω_i. Donaldson (2006) asks whether the underlying compact manifold is always hyperKähler.

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- Multi-moment maps are defined $\nu\colon (M,c) \to \mathcal{P}_{\mathfrak{g}}^*$, where $\mathcal{P}_{\mathfrak{g}} = \ker([\cdot,\cdot]\colon \Lambda^2 \mathfrak{g} \to \mathfrak{g})$.

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- $(2,3)$-trivial Lie algebras may be classified in small dimensions and described and as certain one-dimensional solvable extensions of nilpotent algebras in general.
- G_2 holonomy manifolds with T^2-symmetry correspond via multi-moment map reduction to conformal data on M^4 defined by a certain type of triple of symplectic forms.

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