

HYPERKÄHLER MODIFICATIONS, TWISTS AND IMPLOSIONS

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Some joint work with
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Asian J. Math. 2006
Rev. Sem. Math. Tohoku 2010
Duke Math. J. 2010
+ in progress

HYPERKÄHLER MODIFICATIONS

M^{An} hyper Kähler

metric g
symplectic forms $\omega_S, \omega_T, \omega_A$
 $IS = K = -JI$

X tri-Hamiltonian
isometry
generating an S^1 -action

$L_X g = 0, L_X \omega_A = 0$
 $d\mu_A = X \lrcorner \omega_A$

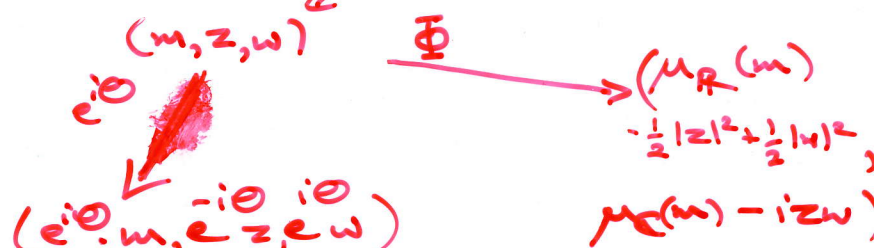


$$M_{\text{mod}} = M \times \mathbb{H} // S^1$$

modification
at level $\epsilon \in \text{Im} \mathbb{H}$

$$= \bar{\Phi}^{-1}(\epsilon) / S^1$$

$$\leftarrow M \times \mathbb{H} = M \times \mathbb{C}^2$$



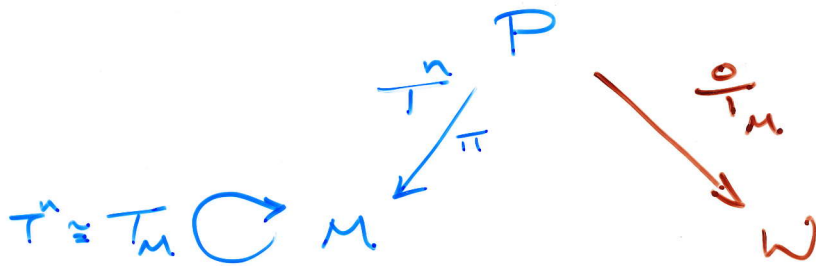
Get

$$b_2(M_{\text{mod}}) = b_2(M) + 1$$



TWIST CONSTRUCTIONS

"GEOMETRIC T-DUALITY"



Given $F_1, \dots, F_n \in \Omega_{\mathbb{Z}}^2(M)$ with $\sum X_i \lrcorner F_i = 0$
 and $X_i \lrcorner F_j = -da_{ij}$ where $a_{ij} \in C^\infty(M)$ and $X_i \in \mathfrak{X}(M)$

there is a T^n -bundle P with connection 1-forms Θ_j , $d\Theta_j = \pi^* F_j$

Lashof-May-Segal

and a lift $\overset{\circ}{T}_M$ to a torus action commuting with the principal action. $X_i = \tilde{X}_i + a_{ij} Y_j$
 ↑ horizontal lift

$W = P / \overset{\circ}{T}_M$ is the twist of M w.r.t. T_M, F & a

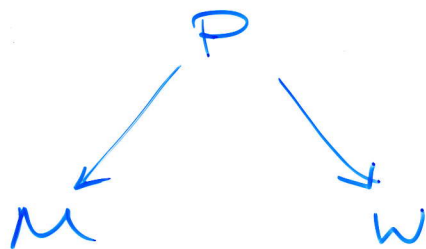
Choosing α so that $\overset{\circ}{T}_M$ is transverse to $\mathfrak{h} = \ker \Theta$ we may transfer T_M -invariant tensors α from M to α_W on W via

$$\alpha \sim_{\mathfrak{h}} \alpha_W \iff \pi^* \alpha = \pi_W^* \alpha_W \text{ on } \mathfrak{h}$$

Exterior derivatives are related by

$$d\alpha_W \sim_{\mathfrak{h}} d\alpha - (a^{-1})_{ij} F_i \wedge (X_j \lrcorner \alpha)$$

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$$d_W \sim d - \bar{a}' F \wedge X \downarrow$$

• T_M action not assumed free

• W an orbifold in general

Twist construction is good at producing geometries with torsion on W compact, smooth & $\pi_1 = 0$

(a) SKT geometry

(g, I) Hermitian

Bismut connection $\nabla = \nabla^{LE} + \frac{1}{2} \epsilon$, $\epsilon = -I d\omega_I$

$$+ \partial_{\bar{I}} \bar{\partial}_I \omega_I = 0$$

E.g. $M = N \times T^{2n}$

N Kähler

$$F_i \in \Omega_{\mathbb{Z}}^{1,1}(N)$$

"instantons" (or SKT)

linearly independent & $\sum \delta_{ij} F_i \wedge F_j = 0$
for some $(\delta_{ij}) > 0$

e.g. generated by blow-ups.

gives SKT structure ($\epsilon \neq 0$) on W a T^{2n} -bundle over N

cf. Goldstein-Prokushkin, Gratchev-Gratichov-Poon.

(b) HKT geometry

(g, I, J, K)

$$I d\omega_I = J d\omega_J = K d\omega_K$$

\Rightarrow hypercomplex

(i) $M = N \times \mathbb{R}$

HKT products

"instantons" $F_i \in S_{\mathbb{Z}}^2 E(N) = \bigcap_{A=I, J, K} \Omega_{\mathbb{Z}}^{1,1, A}(N)$, \mathbb{R} with symmetries

Relevant for:
string theory + fluxes,
1d quantum mechanics
with type B SWFs.

M. HKT, F_i instantons

twists to

W an R -bundle over N

that is HKT $\pi_1(W) = 0$ & compact from

E.g. N compact hyperKähler
or squashed $S \times S^1$, S 3-Sasakii

$R = T^{4n}$, G compact Lie dim $4n \dots$
or $S \times S^1$ (Joyce)

(ii) W^{4n} compact, $\pi_1 = 0$, HKT
with Obata holonomy in $SL(n, \mathbb{H})$

(iii) W^{4n} compact, $\pi_1 = 0$, hypercomplex
but with no compatible HKT metric
(from non-instanton twist)

Ⓒ hyperKähler geometry

(i) Gibbons - Papadopoulos - Stelle

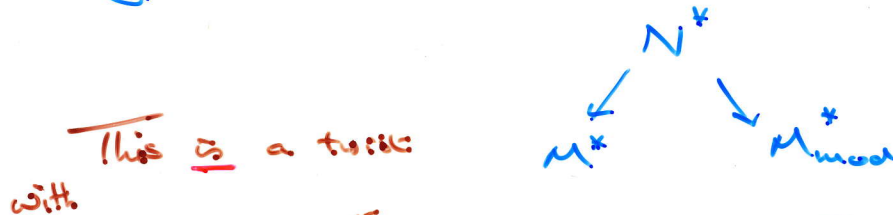
M^4 HK
+ S^1 -symmetry
X

\longleftrightarrow
T-duality
 \parallel
twist
 $F = dX^b$
 $a = \|X\|^2$

W^4 SHKT
Callan-Harvey
-Strominger Ansatz

(ii) hyperKähler modification

$M_{mod} = M \times \mathbb{H} // S^1$
($\varepsilon = 0$)



with
$$F = \frac{2}{\|x\|^3} \left[G_{i,j,k} M^i X^j \wedge X^k + \sum_{i,j,k} M^i X^j \wedge X^k \right] - \frac{1}{\|x\|} dX^b$$

$$a = - \frac{\|x\|^2}{\|x\|} - 1 \quad \text{of} \quad g = g_{HK} + \frac{1}{\|x\|} (X^b)^2_{\mathbb{H}}$$

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CAN WE EXTEND THE ABOVE TO COMPACT NON-ABELIAN G ?

For hyperKähler modifications can try
 M hK with G_M -action
 E hK with $G_L \times G_R$ -action

$$M_{\text{mod}} := M \times E // \Delta G$$
$$\Delta G = \{(g, g) \in G_M \times G_L\}$$

For $G_{M,L,R} = U(n)$, two obvious choices

(a) $E = \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) + \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)^*$
 \rightarrow complicated topology of M_{mod}

(b) $E = T^*G_L(n, \mathbb{C})$
 $= \{ T_1, T_2, T_3 : [0, 1] \rightarrow \mathbb{C} \mid \frac{dT_i}{dt} = \epsilon_{ijk} [T_j, T_k] \} / \mathbb{C}^0$

\rightarrow metric deformations of M .

For $G_{M,L} = SU(2)$, $G_R = U(1)$

$$E = \mathbb{H}^2 = (\mathbb{C} \rightleftarrows \mathbb{C}^2)$$

M_{mod} is hyperKähler with a $U(1)$ -action

such that $\forall \underline{\epsilon} \in \text{Im } \mathbb{H}$

$$M_{\text{mod}} //_{\underline{\epsilon}} U(1) \cong M //_{\underline{\epsilon}} SU(2)$$

where $M //_{\underline{\epsilon}} SU(2) := M \times \tilde{M}(\underline{\epsilon}) // SU(2)$
 $\tilde{M}(\underline{\epsilon}) \cong \begin{cases} T^*(\mathbb{C}P^1) & \underline{\epsilon} \neq 0 \\ \text{nilpotent variety of } \mathfrak{sl}(2, \mathbb{C}) & \underline{\epsilon} = 0 \end{cases}$
 Biquard space

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DEFINITION : For G compact Lie, M hyper-Kähler with triholomorphic G -action a (big) hyper-Kähler implosion of M is a hyper-Kähler space M_{HK_i} with tri-holomorphic T -action s.t.

$$M \underset{\varepsilon}{\underset{\cong}{\parallel}} G = M_{HK_i} \underset{\varepsilon}{\underset{\cong}{\parallel}} T$$

T maximal torus

$\forall \varepsilon \in \mathfrak{t}^*$ in \mathbb{H}

A universal hyper-Kähler implosion for G is a hyper-Kähler space $\mathcal{U}_{HK_i}(G)$ with $G \times T$ -action such that

$$M_{HK_i} = M \times \mathcal{U}_{HK_i}(G) \underset{\cong}{\parallel} \Delta G$$

Examples: ① $\mathcal{U}_{HK_i}(SU(2)) = \mathbb{H}^2$

② $\mathcal{U}_{HK_i}(SU(3)) = (\mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^3) \underset{\cong}{\parallel} SU(2)$

$$SU(2) \leq U(1) \times U(2)$$

$$U(1) \times U(2) / SU(2) = T^2 \cong T^2 \leq SU(3)$$

$$= \mathbb{H}^8 \underset{\cong}{\parallel} SU(2) = \{0\} \cup \mathcal{U}(\tilde{G}_4(\mathbb{R}^8))$$

Theorem: $\mathcal{U}_{HK_i}^{reg}(SU(n)) = \coprod \underline{M}_n^0$

$$\underline{n} = (0 < n_1 < n_2 < \dots < n_r = n)$$

$$C(\underline{\varepsilon}) = C(\varepsilon_2, \varepsilon_3)$$

$$\underline{M}_n^0 = (\mathbb{C}^{n_1} \rightleftarrows \mathbb{C}^{n_2} \rightleftarrows \dots \rightleftarrows \mathbb{C}^n)$$

$$\underset{\cong}{\parallel} SU(n_1) \times SU(n_2) \times \dots \times SU(n_{r-1})$$

$$\cong (SK(n, \mathbb{C}) \times_{[P, P]} [P, P]^0) \quad \text{Parabolic}$$

For general compact G , expect to use $\frac{1}{p} T^*G \underset{\cong}{\parallel} [P, P]$