HyperKähler Modifications, Twists and Implosions

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some joint work with
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HyperKähler Modifications

M^{4n} hyperKähler — metric g, symplectic forms $\omega_I, \omega_J, \omega_K$, $IJ = K = -JI$

X tri-Hamiltonian isometry generating an S^1-action — $L_X g = 0$, $L_X \omega_A = 0$, $d\mu_A = X \lrcorner \omega_A$

$$\Downarrow$$

$$M_{mod} = M \times \mathbb{H} /\!/\!/ S^1 = \Phi^{-1}(\varepsilon)/S^1$$

modification at level $\varepsilon \in \operatorname{Im}\mathbb{H}$

$M \times \mathbb{H} = M \times \mathbb{C}^2$

$$(m, z, w) \xrightarrow{\Phi} \left(\mu_{\mathbb{R}}(m) - \tfrac{1}{2}|z|^2 + \tfrac{1}{2}|w|^2,\ \mu_{\mathbb{C}}(m) - izw\right)$$

$$e^{i\theta}: (m, z, w) \mapsto (e^{i\theta}\cdot m, e^{-i\theta} z, e^{i\theta} w)$$

Get

$b_2(M_{mod}) = b_2(M) + 1$

$N^* = M \times \mathbb{H} \setminus \mu^{-1}(\varepsilon) \times \underline{0}$

$$\begin{tikzcd} & N^* \arrow[dl, "S^1"'] \arrow[dr, "S^1"] & \\ M^* & & M^*_{mod} \end{tikzcd}$$

Twist Constructions

"Geometric T-duality"

$$\begin{tikzcd} & P \arrow[dl, "T^n"', "\pi"] \arrow[dr, "\mathring{T}_M"] & \\ T^n \cong T_M \circlearrowright M & & W \end{tikzcd}$$

Given $F_1, \dots, F_n \in \Omega^2_{\mathbb{Z}}(M)$ with $L_{X_i} F_j = 0$

and

$$X_i \lrcorner F_j = -d a_{ij} \qquad a_{ij} \in C^\infty(M)$$
$$X_i \in \operatorname{Lie} T_M$$

there is a T^n-bundle P with connection 1-forms θ_j, $\quad d\theta_j = \pi^* F_j$.

Lashof-May-Segal

and a lift $\mathring{T}_M$ $\qquad \mathring{X}_i = \tilde{X}_i + a_{ij} Y_j$

to a torus action

commuting with the principal action.

horizontal lift

$$W = P / \mathring{T}_M \quad \text{is the twist of}$$

M w.r.t. T_M, F & a

Choosing a so that $\mathring{T}_M$ is transverse to $\mathcal{H} = \ker \theta$ we may transfer T_M-invariant tensors α from M to α_W on W via

$$\alpha \sim_{\mathcal{H}} \alpha_W \iff \pi^* \alpha = \pi_W^* \alpha_W \text{ on } \mathcal{H}$$

Exterior derivatives are related by

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - (a^{-1})_{ij} F_i \wedge (X_j \lrcorner \alpha)$$

③

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ M & & W \end{tikzcd}$$

$d_W \sim_{\mathcal{H}} d - a^{-1} F \wedge X \lrcorner$

- T_M action not assumed free
- W an orbifold in general

Twist construction is good at producing geometries with torsion on W compact, smooth & $\pi_1 = 0$

Relevant for: string theory + fluxes, 1d quantum mechanics with type B SUSYs.

ⓐ SKT geometry

(g, I) Hermitian

Bismut connection $\nabla = \nabla^{LC} + \frac{1}{2} c$, $c = -I d\omega_I$

$+ \quad \partial_I \bar{\partial}_I \omega_I = 0$

E.g. $M = N \times T^{2n}$ $\quad$ N Kähler (or SKT)

$F_i \in \Omega^{1,1}_{\mathbb{Z}}(N)$ "instantons"

linearly independent & $\sum \gamma_{ij} F_i \wedge F_j = 0$ for some $(\gamma_{ij}) > 0$

e.g. generated by blow-ups.

gives SKT structures $(c \neq 0)$ on W a T^{2n}-bundle over N

cf. Goldstein–Prokushkin, Grantcharov–Grantcharov–Poon.

ⓑ HKT geometry

(g, I, J, K) $\quad$ $I d\omega_I = J d\omega_J = K d\omega_K$

$\Rightarrow$ hypercomplex

(i) $M = N \times R$ $\quad$ HKT product

"instantons" $F_i \in S^2_{\mathbb{Z}} E(N) = \bigcap_{A = I, J, K} \Omega^{1,1,A}_{\mathbb{Z}}(N)$, R with symmetries

④

M HKT, F_i instantons

twists to

W an R-bundle over N

that is HKT $\quad \pi_1(W) = 0$ & compact from

E.g. N compact hyperKähler

or squashed $S \times S^1$, S 3-Sasaki

$R = T^{4n}$, G compact Lie dim $4n \ldots$ (Joyce)

or $S \times S^1$

(ii) W^{4n} compact, $\pi_1 = 0$, HKT

with Obata holonomy in $SL(n, \mathbb{H})$

(iii) W^{4n} compact, $\pi_1 = 0$, hypercomplex

but with no compatible HKT metric

(from non-instanton twist)

(c) hyperKähler geometry

(i) Gibbons - Papadopoulos - Stella

M^4 hK $\quad \longleftrightarrow \quad W^4$ SHKT

$+ S^1$-symmetry $\quad$ T-duality $\quad$ Callan-Harvey-Strominger Ansatz

X $\quad$ $\|$ twist

$F = dX^\flat$

$a = \|X\|^2$

(ii) hyperKähler modification $\quad M_{mod} = M \times \mathbb{H} /\!/\!/ S^1$

$(\varepsilon = 0)$

$$\begin{array}{ccc} & N^* & \\ \swarrow & & \searrow \\ M^* & & M^*_{mod} \end{array}$$

This is a twist

with

$$F = \frac{2}{\|\mu\|^3}\Big[\mathfrak{S}_{I,J,K}\, \mu^I\, JX^\flat \wedge KX^\flat + \sum_{I,J,K} \mu^I\, IX^\flat \wedge X^\flat\Big] - \frac{1}{\|\mu\|}\, dX^\flat$$

$$a = -\frac{\|X\|^2}{\|\mu\|} - 1 \quad \text{of} \quad g = g_{hK} + \frac{1}{\|\mu\|}(X^\flat)^2_{\mathbb{H}}$$

CAN WE EXTEND THE ABOVE TO COMPACT NON-ABELIAN G ?

For hyperKähler modifications can try

M hK with G_M-action

E hK with $G_L \times G_R$-action

$$M_{mod} := M \times E /// \Delta G$$

$$\Delta G = \{(g,g) \in G_M \times G_L\}$$

For $G_{M,L,R} = U(n)$, two obvious choices

a) $E = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) + \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)^*$

$\longrightarrow$ complicated topology of M_{mod}

b) $E = T^* GL(n, \mathbb{C})$

$$= \left\{ T_1, T_2, T_3 : [0,1] \longrightarrow \mathfrak{g} \;\middle|\; \frac{dT_i}{dt} = \epsilon_{ijk}[T_j, T_k] \right\} \Big/ \mathcal{G}_0^0$$

$\longrightarrow$ metric deformations of M.

For $G_{M,L} = SU(2)$, $G_R = U(1)$

$$E = \mathbb{H}^2 = (\mathbb{C} \rightleftarrows \mathbb{C}^2)$$

M_{mod} is hyperKähler with a $U(1)$-action such that $\forall\, \underline{\varepsilon} \in \mathrm{Im}\,\mathbb{H}$

$$M_{mod} ///_{\underline{\varepsilon}} U(1) = M \,\widetilde{///}_{\underline{\varepsilon}}\, SU(2)$$

where $M \,\widetilde{///}_{\underline{\varepsilon}}\, SU(2) := M \times \widetilde{M}(\underline{\varepsilon}) /// SU(2)$

$$\widetilde{M}(\underline{\varepsilon}) \cong \begin{cases} T^*\mathbb{CP}(1) & \underline{\varepsilon} \neq \underline{0} \\ \text{nilpotent variety of } sl(2,\mathbb{C}) & \underline{\varepsilon} = \underline{0} \end{cases}$$

Biquard space

Definition: For G compact Lie, M hyperKähler with triholomorphic G-action a (big) hyperKähler implosion of M is a hyperKähler space M_{hKi} with tri-holomorphic T-action s.t.

T maximal torus

$$M \,\widetilde{/\!/\!/}_{\underline{\varepsilon}}\, G = M_{hKi} /\!/\!/_{\underline{\varepsilon}} T$$

$\forall \underline{\varepsilon} \in \mathfrak{t}^*_{\oplus} \text{ in } \mathbb{H}$

A universal hyperKähler implosion for G is a hyperKähler space $U_{hKi}(G)$ with $G \times T$-action such that

$$M_{hKi} = M \times U_{hKi}(G) /\!/\!/ \Delta G$$

Examples: ① $U_{hKi}(SU(2)) = \mathbb{H}^2$

② $U_{hKi}(SU(3)) = \left(\mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \mathbb{C}^3\right) /\!/\!/_{SU(2)}$

$SU(2) \leq U(1) \times U(2)$

$U(1) \times U(2) / SU(2) = T^2 \cong T^2 \leq SU(3)$

$= \mathbb{H}^8 /\!/\!/ SU(2) = \{0\} \cup \mathcal{U}(\widetilde{Gr}_4(\mathbb{R}^8))$

Theorem: $U^{reg}_{hKi}(SU(n)) = \coprod M^{\circ}_{\underline{n}}$

$\underline{n} = (0 < n_1 < n_2 < \cdots < n_r = n)$

$C(\underline{\varepsilon}) = C(\varepsilon_2, \varepsilon_3)$

$$M^{\circ}_{\underline{n}} = \left(\mathbb{C}^{n_1} \rightleftarrows \mathbb{C}^{n_2} \rightleftarrows \cdots \rightleftarrows \mathbb{C}^{n}\right)$$

$$/\!/\!/ SU(n_1) \times SU(n_2) \times \cdots \times SU(n_{r-1})$$

$$\cong \left(SL(n, \mathbb{C}) \times_{[P,P]} [P,P]^{\circ}\right)_{\circ}$$

P parabolic

For general compact G, expect to use $\coprod_P T^* G^{\mathbb{C}} /\!/^{\mathbb{C}} [P,P]$