Nearly Kähler manifolds with torus symmetry

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Joint work with Giovanni Russo

Russo, G. and Swann, A. F. (2019), 'Nearly Kähler six-manifolds with two-torus symmetry', *J. Geom. Phys.* 138: 144–53
Russo, G. (2021), 'Multi-moment maps on nearly Kähler six-manifolds', *Geom. Dedicata*, 213: 57–81

Nearly Kähler geometry

A *(strict) nearly Kähler manifold* is an almost Hermitian manifold (M, g, J) such that

$$(\nabla_X J)X = 0 \quad \text{and} \quad \nabla J \neq 0.$$

Introduced and extensively studied by Gray (1965) and subsequent papers.

Nagy (2002): complete, simply-connected nearly Kähler manifolds are products of

- Kähler manifolds,
- three-symmetric spaces,
- twistor spaces of positive quaternionic Kähler manifolds, and/or
- nearly Kähler six-manifolds.

Dimension 6

Nearly Kähler in dimension 6

- are positive Einstein (Gray 1976), so complete examples are compact with π_1 finite
- homogeneous examples are three-symmetric spaces:

$$S^6 = \frac{G_2}{SU(3)}, \quad \mathbb{C}P(3) = \frac{Sp(2)}{Sp(1)U(1)},$$
$$F_{1,2}(\mathbb{C}^3) = \frac{SU(3)}{T^2}, \quad S^3 \times S^3 = \frac{SU(2)^3}{SU(2)_\Delta};$$

  constructed by Wolf and Gray (1968), classified by Butruille (2005)
- admit Killing spinors and their cones are of holonomy G_2
- Foscolo and Haskins (2017) new compact examples: cohomogeneity one on $S^3 \times S^3$ and S^6, with principal orbit $(SU(2) \times SU(2))/U(1)_\Delta = S^2 \times S^3$

Symmetry rank

Connected automorphism groups G

Space	S^6	$\mathbb{C}P(3)$	$F_{1,2}(\mathbb{C}^3)$	$S^3 \times S^3$
3-symmetric	G_2	$\mathrm{Sp}(2)$	$\mathrm{SU}(3)$	$\mathrm{SU}(2)^3$
Cohom. 1	$\mathrm{SU}(2)^2$			$\mathrm{SU}(2)^2$

Observe $\operatorname{rank} G \geqslant 2$
with $\operatorname{rank} G > 2$ only for the three-symmetric structure on $S^3 \times S^3$

Moroianu and Nagy (2019) For six-dimensional nearly Kähler manifolds, the connected automorphism group G satisfies $\operatorname{rank} G \leqslant 3$

Aim study nearly Kähler six-manifolds with an effective action of T^2

Differential forms

(M^6, g, J) nearly Kähler with 2-form $\sigma = g(J\,\cdot\,, \,\cdot\,)$.
Put

$$\psi_+ = \frac{1}{3} d\sigma \quad \text{and} \quad \psi_- = \psi_+(J\,\cdot\,, J\,\cdot\,, J\,\cdot\,).$$

$\psi_+ + i\psi_- \in \Lambda^{3,0}$ and is of constant length. The structure group reduces to $SU(3)$.
Can rescale g, so

$$d\psi_- = -2\sigma \wedge \sigma.$$

These forms are preserved by the symmetries of (M, g, J).

$$d\sigma = 3\psi_+$$

is closed, and exact, with invariant primitive.

g is determined by σ and ψ_+ via

$$g(X, Y)\sigma^3 = 3(X \lrcorner \psi_+) \wedge (Y \lrcorner \psi_+) \wedge \sigma.$$

Abelian multi-moment maps

If T^k acts on preserving a form $\alpha \in \Omega^r(M)$, then

$$\nu \colon M \to \Lambda^r \operatorname{Lie}(T^k)^*$$
$$\nu(X_1 \wedge \cdots \wedge X_r) = \alpha(X_1, \dots, X_r)$$

is a *multi-moment map* for the action.

This generalises the idea of Abelian moment map when $d\alpha = \omega$ is a symplectic form, since

$$d(\nu(X_1 \wedge \cdots \wedge X_r))(\,\cdot\,) = (d\alpha)(X_1, \dots, X_r, \,\cdot\,)$$

For (M^6, g, J) nearly Kähler with T^2-symmetry generated by U, V, we call

$$\nu \colon M \to \mathbb{R} = \Lambda^2 \operatorname{Lie}(T^2)^* \qquad \nu = \sigma(U, V)$$

the multi-moment map.

First properties

$$\nu\colon M \to \mathbb{R} \qquad \nu = \sigma(U,V)$$

Satisfies

$$\Delta\nu = 24\nu$$

and

$$d\nu = 3\psi_+(U,V,\,\cdot\,)$$

Proposition

For (M^6,g,J) *complete and connected,* $\nu(M) = [a,b]$ *is a compact interval containing* 0 *in its interior.*

In particular, ν has regular values in $[a,b]$.

Reduction at regular values

$s \neq 0$ a regular value of $\nu = \sigma(U, V)$.
Connection one-forms ϑ_1, ϑ_2 dual to U, V and zero on $\mathrm{Span}\{U, V\}^{\perp}$. Get three one-forms

$$\alpha_0 = \psi_-(U, V, \cdot), \quad \alpha_1 = s\vartheta_1 + V \lrcorner \sigma, \quad \alpha_2 = s\vartheta_2 - U \lrcorner \sigma$$

that descend to $Q = \nu^{-1}(s)/T^2$.

Nearly Kähler metric

$$g = \frac{1}{9(h^2 - s^2)} ds^2 + \vartheta^T H \vartheta + \frac{1}{h^2 - s^2}(\alpha_0^2 + \alpha^T H \alpha)$$

for $\vartheta = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}$, $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $H = \begin{pmatrix} g(U,U) & g(U,V) \\ g(U,V) & g(V,V) \end{pmatrix}$ and $h^2 = \det H$.

RECONSTRUCTION

THEOREM

One-forms α_0, α_1, α_2 on Q^3 satisfying

$$d\alpha_0 = f\,\alpha_1 \wedge \alpha_2, \quad d(f\alpha_1) \wedge \alpha_0 = 0 = d(f\alpha_2) \wedge \alpha_0$$

together with a choice of metric H on $\mathrm{Span}\{\alpha_1, \alpha_2\}$ determine a nearly Kähler manifold with T^2-symmetry via a geometric flow.

If Q is homogeneous with invariant data, then Q is any non-Abelian unimodular group. The flow is

$$\alpha_0' = \frac{4s}{3(h^2 - s^2)}\alpha_0, \quad \alpha' = \frac{s}{3}\left(\frac{8}{h^2 - s^2}1_2 - \frac{1}{h^2}PH\right)\alpha,$$

$$H' = -\frac{1}{s}H + \frac{h^2 - s^2}{3sh^2}HPH, \qquad \text{where } d\alpha = \alpha_0 \wedge P\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\alpha.$$

Can be diagonalised.

Example

$Q = \mathrm{SO}(3) \qquad \mathfrak{so}(3)^* = (23, 31, 12)$.

Particular solution. For

$$s = -\frac{2}{3\sqrt{3}}\cos(2t) \qquad \text{with } t \in (0, \pi/2)$$

have

$$\begin{aligned}
\alpha_0 &= \frac{4}{27}\sin(2t)e_1, \\
\alpha_1 + \alpha_2 &= -\frac{4}{3\sqrt{3}}\frac{\sin(t)\sin(2t)}{2 - \cos(2t)}e_2, \\
\alpha_1 - \alpha_2 &= -\frac{4}{3\sqrt{3}}\frac{\cos(t)\sin(2t)}{2 + \cos(2t)}e_3.
\end{aligned}$$

Gives cohomogeneity-one action of $T^2 \times \mathrm{SU}(2)$ on $S^3 \times S^3$.
Action is missing from Podestà and Spiro (2010).
Should be in the cohomogeneity two results of Madnick (2020).

Stabilisers

(M^6, g, J) nearly Kähler with T^2-symmetry, $\nu = \sigma(U, V)$. Recall $\nu(M) = [a, b]$ with $0 \in (a, b)$.
Stabilisers are either:

- T^2,
- circle subgroups, or
- finite cyclic subgroups.

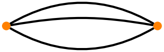
The first two only occur in $\nu^{-1}(0)$, the last can only occur outside $\nu^{-1}(0)$.
$\nu^{-1}(0)/T^2$ is a topological 3-manifold containing a trivalent graph with

- points corresponding to stabiliser T^2 and
- edges corresponding to stabiliser a circle.

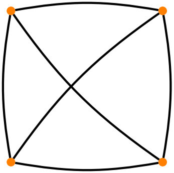
Known graphs for T^2-actions

In $\nu^{-1}(0)/T^2$

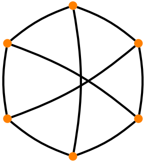
S^6



$\mathbb{C}P(3)$



$F_{1,2}(\mathbb{C}^3)$



$S^3 \times S^3$

$\varnothing$,

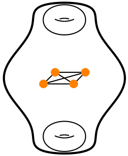
, or

Critical sets at non-zero ν

Occur if and only if U, V linearly dependent over $\mathbb{C}$ but not over $\mathbb{R}$.

Same condition for points with finite stabiliser.

For three-symmetric S^6, $\mathbb{C}\mathrm{P}(3)$ and $F_{1,2}(\mathbb{C}^3)$ only two such sets, from maximum and minimum of ν. Both are pseudo-holomorphic tori, and $\min \nu = -\max \nu$.



For $S^3 \times S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)_\Delta$ different T^2 have different behaviours; can have $\min \nu \neq -\max \nu$, saddle points or 4-dimensional critical sets.

At critical points the Hessian of ν satisfies

$$\mathrm{Hess}(X, Y) + \mathrm{Hess}(JX, JY) + 12\nu g^{\perp}(X, Y) = 0$$

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