Modifying hyperKähler manifolds

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March 2006 / Kühlungsborn

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Joint work with Andrew Dancer
arXiv:math.DG/0510501

Outline

1. **Symplectic Cuts**
   - Lerman's Construction

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   - Lerman's Construction
2. **HyperKähler Modifications**
   - HyperKähler Moment Maps
   - Modifications
   - Topology
   - Examples
3. **Related Constructions**
   - Three-Sasakian
   - Hypersymplectic

Outline

Lerman's Symplectic Cut

- M a symplectic manifold with Hamiltonian circle action.
- Moment map $\mu\colon M \to \mathbb{R}$ satisfies $d\mu = \xi \lrcorner \omega$.
- Level $\varepsilon \in \mu(M)$ with the S^1-action free on $M_\varepsilon = \mu^{-1}(\varepsilon)$.

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$M /\!/ S^1 = M_\varepsilon / S^1$

is a smooth manifold of dimension $\dim M - 2$

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Symplectic cut

$$M_{\mathrm{cut}} = M_{>\varepsilon} \coprod M /\!/ S^1$$

where $M_{>\varepsilon} = \mu^{-1}(\varepsilon, \infty)$
$\dim M_{\mathrm{cut}} = \dim M$

Constructing Symplectic Cuts

$M \times \mathbb{C}$ has

- S^1-action $e^{\mathbf{i}\theta}(m, z) = (e^{\mathbf{i}\theta} m, e^{-\mathbf{i}\theta} z)$,
- moment map $\Phi(m, z) = \mu(m) - |z|^2$.

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$\Phi^{-1}(\varepsilon) = \Sigma_1 \coprod \Sigma_2$

- $\Sigma_1 = \{(m,z) : \mu(m) > \varepsilon,$
  $|z| = +\sqrt{\mu(m) - \varepsilon} > 0\}$
- $\Sigma_2 = \{(m,0) : \mu(m) = \varepsilon\}$

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- $M_{>\varepsilon} = \Sigma_1 / S^1$
- $M /\!/ S^1 = \Sigma_2 / S^1$

Outline

HyperKähler Manifolds

M hyperKähler

- complex structures I, J, K with

$$I^2 = -1 = J^2 = K^2, \qquad IJ = K = -JI,$$

- metric g Hermitian with respect to I, J, K

$$g(IX, IY) = g(X, Y) \qquad \text{etc.}$$

- two forms $\omega_I = g(I\cdot, \cdot)$, etc. symplectic

$$d\omega_I = 0 = d\omega_J = d\omega_K.$$

HyperKähler Moment Maps

(M, g, I, J, K) hyperKähler preserved by S^1-action generated by ξ.

Definition

$\mu = (\mu_I, \mu_J, \mu_K) \colon M \to \mathbb{R}^3$ is a *hyperKähler moment map* if

$$d\mu = (d\mu_I,\ d\mu_J,\ d\mu_K) = (\xi \lrcorner \omega_I,\ \xi \lrcorner \omega_J,\ \xi \lrcorner \omega_K)).$$

The S^1-action is then *tri-Hamiltonian*.

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For flat $M = \mathbb{H} = \mathbb{C} + \mathbf{j}\mathbb{C}$ with

$$q \mapsto e^{\mathbf{i}\theta} q, \qquad \mu^{\mathbb{H}}(q) = \tfrac{1}{2} q\mathbf{i}\bar{q},$$

$$(z, w) \mapsto (e^{\mathbf{i}\theta} z, e^{-\mathbf{i}\theta} w), \qquad \mu^{\mathbb{H}}(z, w) = (\tfrac{1}{2}(|z|^2 - |w|^2),\ \mathbf{i}zw).$$

The HyperKähler Quotient

M hyperKähler with tri-Hamiltonian S^1-action

A level ε is *good* if S^1 acts freely on $\mu^{-1}(\varepsilon) \subset M$.

Theorem (Hitchin-Karlhede-Lindström-Roček)

$M /// S^1$ is a hyperKähler manifold when ε is good.

$\dim(M /// S^1) = \dim M - 4$

When M is complete so is $M /// S^1$.

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HyperKähler Modifications

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Let ε be a good level for μ.

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HyperKähler modification

is the hyperKähler quotient of $M \times \mathbb{H}$ at level ε:

$$M_{\mathrm{mod}} = \Phi^{-1}(\varepsilon)/S^1$$

Properties of the Modification

M_{mod} is

- a smooth hyperKähler manifold of the same dimension as M,
- complete if M is complete.

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The modification construction can be iterated.

Symplectic vs. HyperKähler

Symplectic Cut

$z \mapsto |z|^2$

$\mathbb{C} \to \mathbb{R}$

HyperKähler Modification

$q \mapsto q\mathbf{i}\bar{q}$

$\mathbb{H} \to \mathbb{R}^3$

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image $\mathbb{R}_{\geqslant 0} \subsetneq \mathbb{R}$
section $p \mapsto +\sqrt{p}$

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surjective
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$\Phi\colon M \times \mathbb{C} \to \mathbb{R}$
$\Phi^{-1}(\varepsilon) \to M$ image $M_{\geqslant \varepsilon}$

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$\Phi \colon M \times \mathbb{H} \to \mathbb{R}^3$
$N := \Phi^{-1}(\varepsilon) \to M$ onto
$M_{\text{mod}} = M^*_{\text{mod}} \coprod M /\!/\!/ S^1$

$$M^* \leftarrow N^* \rightarrow M^*_{\text{mod}}$$

non-trivial circle bundles

Outline

Euler Characteristic

Fixed-point sets are related by

$$(M_{\text{mod}})^{S^1} = (M /\!/\!/ S^1) \coprod M^{S^1}.$$

If the circle action is free away from the fixed-point set and the topological type is finite, then

$$\chi(M_{\text{mod}}) = \chi(M_{\text{mod}}^{S^1}) = \chi(M /\!/\!/ S^1) + \chi(M).$$

Betti Numbers

Theorem

For M hyperKähler with tri-Hamiltonian S^1-action, finite topological type, simply-connected and ε a good level

$$b_2(M_{\mathrm{mod}}) = b_2(M) + 1, \qquad \pi_1(M_{\mathrm{mod}}) = \{1\}.$$

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2. $b_2(M) = b_2(M \setminus \mu^{-1}(\varepsilon)) - 1$

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2. $b_2(M) = b_2(M \setminus \mu^{-1}(\varepsilon)) - 1$
   - Thom-Gysin sequence for $\mu^{-1}(\varepsilon) \hookrightarrow M$
   - The closed Poincaré dual $\eta_{\mu^{-1}(\varepsilon)} = \mu^* \eta_{\{\varepsilon\}} \in H^3(\mathbb{R}^3) = \{0\}$

Outline

Four Dimensions

Multi-instanton metrics

$M = \mathbb{H}$ with $q \mapsto e^{\mathbf{i}\theta} q$.

For any $\varepsilon \neq 0$, $\mu^{-1}(\varepsilon) \cong S^1$, $M /// S^1 = \{*\}$. M_{mod} has two fixed-points, $M_{\text{mod}} = T^* \mathbb{C}\mathrm{P}(1)$ with Euguchi-Hanson metric.

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Iterate to get the multi-instanton metrics of Gibbons-Hawking

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(dt + A_1\,dx_1 + A_2\,dx_2 + A_3\,dx_3)^2$$
$$V(x) = \sum_i \|x - p_i\|^{-1}, \qquad *_3 dA = dV.$$

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Multi-Taub-NUT metrics are obtained starting from $M = \mathbb{R}^3 \times S^1 = \mathbb{H}/\mathbb{Z}$. Same topology as multi-instanton case.

HyperKähler Toric Manifolds

$M = T^* \mathbb{C}P(2)$ has tri-Hamiltonian T^2-action.
Will always have $b_2(M_{\text{mod}}) = 2$, but for different choices of S^1 in T^2 can get

$$b_4(M_{\text{mod}}) = 2 \quad \text{or} \quad 3.$$

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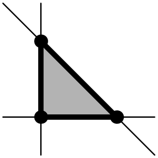
Poincaré polynomial

$$P_t(M) = \sum_{k=0}^{n} d_k (t^2 - 1)^k$$

where d_k is the number of k-dimensional elements in the bounded polyhedral complex.

Toric Modifications

- $T^* \mathbb{C}P(2)$, $b_2 = 1 = b_4$



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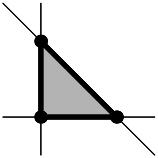

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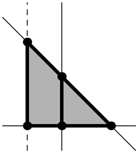

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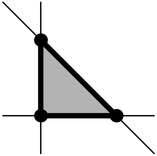

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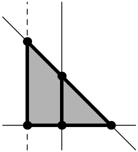

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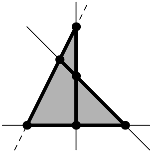

$b_2 = 2$, $b_4 = 3$

Gauge Theory Quotients

$T^*G_{\mathbb{C}}$ for G compact Lie

- is hyperKähler, as it is the moduli space of $\mathfrak{g}$-valued solutions to Nahm's equations on $[0, 1]$
- carries a tri-Hamiltonian action of $G \times G$
- can repeatedly be modified with respect a circle in a maximal torus $\mathbb{T} \subset G$ to get complete hyperKähler metrics with $G \times \mathbb{T}$-symmetry.

Outline

Three-Sasakian Modifications

$(\mathscr{S}, g)$ is three-Sasaki if the cone

$$C(\mathscr{S}) = \mathbb{R}_{>0} \times \mathscr{S}, \qquad dt^2 + t^2 g$$

is hyperKähler.

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The hyperKähler modification of $C(\mathscr{S})$ at level 0 is a cone $C(\mathscr{S}_{\text{mod}})$.

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Proposition

$\mathscr{S}$ is compact with free-circle action on $\mu_{\mathscr{S}}^{-1}(0)$. Then $\mathscr{S}_{\text{mod}}$ is smooth and compact and contains a copy of $\mathscr{S} /// S^1$.
If $\mathscr{S}$ is simply connected, then so is $\mathscr{S}_{\text{mod}}$ and

$$b_2(\mathscr{S}_{\text{mod}}) = b_2(\mathscr{S}) + 1.$$

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Hypersymplectic Manifolds

(M, g, I, S, T) is hypersymplectic if

$$I^2 = -1, \qquad S^2 = +1, \qquad IS = T = -IS,$$
$$g(IX, IY) = g(X, Y), \qquad g(SX, SY) = -g(X, Y),$$
$$d\omega_I = 0 = d\omega_S = d\omega_T, \qquad \omega_A(\cdot, \cdot) = g(A\cdot, \cdot)$$

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Example $\mathbb{C}^{1,1}$ with circle action $(z, w) \mapsto (e^{\mathbf{i}\theta} z, e^{\mathbf{i}\theta} w)$ has moment map

$$\phi_{\mathrm{HS}}(z, w) = (\tfrac{1}{2}(|z|^2 + |w|^2), \mathbf{i} z \bar{w})$$

which descends to a two-to-one map with image the solid cone

$$\{(a, b) \in \mathbb{R} \times \mathbb{C} : a \geqslant |b|\}.$$

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$$\{(a, b) \in \mathbb{R} \times \mathbb{C} : a \geqslant |b|\}.$$

This combines features of the symplectic and hyperKähler cases

- ϕ_{HS} is not onto,
- ϕ_{HS} has no section.

Hypersymplectic Cuts

M hypersymplectic with tri-Hamiltonian circle action moment map $\mu\colon M \to \mathbb{R}^3$, ε a good level.
On $M \times \mathbb{C}^{1,1}$ consider the action $e^{\mathbf{i}\theta}(m, p) = (e^{\mathbf{i}\theta} m, e^{-\mathbf{i}\theta} p)$ with moment map

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$M_{\mathrm{cut}} = X \coprod M^*_{\mathrm{cut}}$, where

- $X = \mu^{-1}(\varepsilon)/S^1$ is the hypersymplectic quotient,
- M^*_{cut} is related to

$$\{m \in M : \mu(m) - \varepsilon = (a,b),\ a > 0,\ a \geqslant |b|\}$$

via non-trivial circle bundles.

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- preserves completeness,
- can be iterated.

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