

Twists, Torsion and T-duality

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Outline

- 1 **Motivation**
 - HKT and String Duals
 - Geometry with Torsion

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2 Twist Constructions

- Basic Construction
- Lifting Actions
- Transformation Rules

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An Example of T-duality

HyperKähler M^4

$$ds^2 = V^{-1}(d\tau + \omega)^2 + V\gamma_{ij}dx^i dx^j$$

$$dV = *_3 d\omega$$

T duality

$$\longleftrightarrow$$

on $X = \frac{\partial}{\partial \tau}$

Strong HKT W^4

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For circle actions have:

$$R \leftrightarrow 1/R \quad \text{and here} \quad W = (M/S^1) \times S^1$$

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Metric geometry with torsion

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$$\nabla = \nabla^{\text{LC}} + \frac{1}{2}c$$

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- ∇ and ∇^{LC} have the same geodesics/dynamics

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Definition

The geometry is *strong* if $dc = 0$

KT Geometry

Metric geometry

$$g, \nabla = \nabla^{\text{LC}} + \frac{1}{2}c, c \in \Lambda^3 T^*M$$

KT geometry

additionally

- I integrable complex structure
- $g(IX, IY) = g(X, Y)$
- $\nabla I = 0$

Here $I: TM \rightarrow TM$ with

$$I^2 = -1 \quad N_I = 0$$

where $N_I(X, Y) =$

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Given (g, I) the connection ∇ is

unique: $c = -IdF_I$, where

$$F_I(X, Y) = g(IX, Y)$$

- KT geometry is just Hermitian geometry together with the Bismut connection ∇
- $c = 0$ is Kähler geometry
- strong KT geometry is $\partial\bar{\partial}F_I = 0$
- Gauduchon, 1991: every compact Hermitian M^4 is conformal to strong KT

HKT geometry

HKT structure

(g, ∇, I, J, K) such that

- each (g, ∇, A) is KT, $A = I, J, K$
- $IJ = K = -JI$

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- most commonly encountered hypercomplex structures (M, I, J, K) admit an HKT metric — but not all.
- there is a good potential theory

$$F_I = \frac{1}{2}(1 - J)dId\rho$$

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Example

$G = SU(3) = M^8$,
bi-invariant g is
strong HKT

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Twist

- X generating a circle action on M
- $(P, \theta) \xrightarrow{\pi} M$ an invariant principal S^1 -bundle
- X' a lift of X generating a free circle action

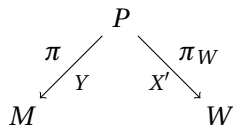
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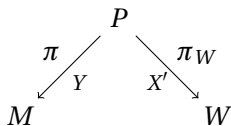
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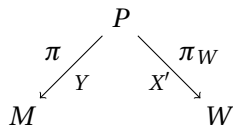
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Dually

M is a twist of W with respect to X_W

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- X a vector field on M
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Lemma

There is an X' on P preserving θ and projecting to X if and only if X^θ is exact.

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Proof.

Let \tilde{X} be the horizontal lift of X .

Then

$$X' = \tilde{X} + aY$$

with $da = -X^\theta$. □

Lifting Circle Actions

Call an S^1 -action generated by X *F-Hamiltonian* if X preserves $F \in \Omega^2(M)$ and $X \lrcorner F$ is exact.

Proposition (cf. Lashof, May, and Segal, 1983)

Suppose $F \in \Omega_{\mathbb{Z}}^2(M)$ is a closed 2-form with integral periods. For each F -Hamiltonian S^1 -action and each principal circle bundle $P \rightarrow M$ with $c_1(P) \otimes \mathbb{R} = [F]$ there is a locally free circle action on P covering the action on M and an invariant principal connection θ such that $F_{\theta} = F$.

General circle actions on $\mathbb{C}P(n)$ can not be lifted to *free* circle actions on P .

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$$\pi_W^* \alpha_W = \pi^* \alpha - \theta \wedge \pi^* \left(\frac{1}{a} X \lrcorner \alpha \right)$$

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- For metrics

$$\pi_W^* g_W = \pi^* g - 2\theta \vee \pi^* \left(\frac{1}{a} X^\flat \right) + \pi^* \left(\frac{1}{a^2} \|X\|^2 \right) \theta^2$$

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Lemma

$d\alpha_W$ is \mathcal{H} -related to a form on M if and only if $L_X \alpha = 0$. Then

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - F_\theta \wedge \frac{1}{a} X \lrcorner \alpha.$$

Almost Hermitian Twist

Definition

Let (M, g, F_I) be an almost Hermitian structure invariant under X . This has *twist* (W, g_W, F_I^W) where

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- *If I is integrable then I_W is integrable if and only if $F_\theta \in \Lambda^{1,1}$*

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Proposition

- If I is integrable then I_W is integrable if and only if $F_\theta \in \Lambda^{1,1}$
- the forms $c = -IdF_I$ are related by

$$c_W \sim_{\mathcal{H}} c - \frac{1}{a} X^\flat \wedge IF_\theta$$

Transformation Rules II

Corollary

If (M, g, I, J, K) is hyperHermitian (resp. HKT) then (W, g_W, I_W, J_W, K_W) is hyperHermitian (resp. HKT) if and only if

$$F_\theta \in \bigcap_{A=I,J,K} \Lambda_A^{1,1}$$

i.e. F_θ is an instanton

Generalises Joyce, 1992, and Grantcharov and Poon, 2000

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Corollary

For M KT (resp. HKT) and F_θ an instanton, W is strong if and only if

$$dc = \frac{1}{a}(dX^b + X \lrcorner c - \frac{1}{a}\|X\|^2 F_\theta) \wedge F_\theta$$

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From a HyperKähler Metric

- $g = \frac{1}{V}\varphi^2 + Vh$
hyperKähler, $c = 0$
- hyperKähler isometry X
 - $\varphi(X) = 1, \quad L_X\varphi = 0$
 - $X^\flat = V^{-1}\varphi, \quad V = \|X\|^{-2}$
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Taking $F_\theta = \lambda dX^\flat \neq 0$ gives an HKT twist if $X \lrcorner F_\theta = -\lambda d\|X\|^2$ is exact, so $\lambda = \lambda(\|X\|^2)$.

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This is a twist via a trivial bundle with non-flat connection.

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- $b_2(\mathbb{CP}(2)) = 1$ generated by $[\omega_{\mathbb{CP}(2)}]$
- P, θ pull-back to $M = \mathcal{U}(\mathbb{CP}(2))/\mathbb{Z}$ of the circle bundle with $F_{\theta} = \pi^* \omega_{\mathbb{CP}(2)}$

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- Topologically $\mathcal{U}(\mathbb{CP}(2)) / \mathbb{Z} = \frac{SU(3)}{U(1)} \times S^1$. The S^1 acts as HKT isometries.
- $b_2(\mathbb{CP}(2)) = 1$ generated by $[\omega_{\mathbb{CP}(2)}]$
- P, θ pull-back to $M = \mathcal{U}(\mathbb{CP}(2)) / \mathbb{Z}$ of the circle bundle with $F_{\theta} = \pi^* \omega_{\mathbb{CP}(2)}$

Twist of $\mathcal{U}(\mathbb{CP}(2)) / \mathbb{Z}$: strong HKT structure on $SU(3)$.

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3 Examples

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Nilmanifold Examples

Theorem (Fino, Parton, and Salamon, 2004)

The six-dimensional strong KT nilmanifolds have Lie algebras

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Mejldal, 2004

The 8-dimensional nilmanifolds with $\mathfrak{g} = (0^6, 13 - 24 + 56, 12 - 2.23 + 3.34)$ are irreducible and lie in a 15-dimensional family of invariant strong KT structures.

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Remark

All known strong KT structures on nilmanifolds may be obtained via iterations of the above twist constructions starting from a flat torus.

Non-toral Base

- Twisting $M^6 = N^4 \times T^2$
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Take N^4 a K3 surface, with Kähler forms $\omega_I, \omega_J, \omega_K$. Integrability,

$$F_1 + iF_2 = \alpha + \lambda_1 \omega_I + \lambda_2 (\omega_J + i\omega_K)$$

with $\alpha \in \Lambda_I^{1,1} \cap (\omega_I)^\perp$. Strong,

$$\alpha \wedge \bar{\alpha} = 4(|\lambda_1|^2 - 2|\lambda_2|^2) \text{vol}_g$$

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Theorem

For linearly independent primitive F_i satisfying the conditions to the left, twist W^6 of $M^6 = N^4 \times T^2$ is a compact simply-connected strong KT manifold.

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- n -torus twists: are governed by $da_{ij} = -X_i \lrcorner F_j$. Wider variety of phenomena.
- multiple twists: are not the same as n -torus twists.

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 - based on a double principal bundle $M \longleftarrow P \longrightarrow W$ with common Ehresmann connection \mathcal{H}
- defining forms are \mathcal{H} -related
- twisting by instantons preserves KT and HKT geometries
- strong structures may be obtained
- non-instanton twists are also necessary

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Exterior derivative of the torsion form

$$\begin{aligned}
 dc_W \sim_{\mathcal{H}} & dc - \frac{1}{a} dX^b \wedge IF_\theta + \frac{1}{a} X^b \wedge d(IF_\theta) \\
 & - F_\theta \wedge \frac{1}{a} X \lrcorner c + F_\theta \wedge \frac{1}{a^2} \|X\|^2 IF_\theta - F_\theta \wedge \frac{1}{a} X^b \wedge X \lrcorner IF_\theta
 \end{aligned}$$