MOMENT MAPS AND THE GEOMETRY OF THREE-FORMS

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Plan

I, Geometry
- 3-forms
- Multi-moment maps
- Topological existence

II, Algebra
- Algebraic existence
- Structure of (2,3)-trivial Lie algebras
- Theory of orbits

III, G_2 geometry
- T^2-symmetry
- Reduction and reconstruction

I, GEOMETRY

Three-forms on vector spaces

$V = \mathbb{R}^n \qquad c \in \Lambda^3 V^*$

<u>Definition</u>: c is <u>fully</u> <u>non</u>-<u>degenerate</u> if

$$c(v, w, \cdot) \neq 0 \qquad \forall \text{ linearly indep. } v, w \in V$$

<u>Theorem (Massey, 1983)</u>: c fully non-degenerate $\Longrightarrow$ $n = 3$ or 7.

<u>Definition</u>: c is (<u>weakly</u>) <u>non</u>-<u>degenerate</u> if

$$c(v, \cdot, \cdot) \neq 0 \qquad \forall\, v \in V,\ v \neq 0.$$

<u>Proposition</u>: c non-degenerate $\Longrightarrow$ $n \neq 1, 2, 4$

<u>Definition</u> (Hitchin): c is <u>stable</u> if $GL(V) \cdot c \subseteq \Lambda^3 V^*$ is open.

<u>Proposition</u>: c stable $\Longrightarrow$ $n = 3, 4, 5, 6, 7, 8$

M^n a manifold of dimension n.

Definition: a strong geometry is a pair (M, c), $c \in \Omega^3(M)$ such that

$$dc = 0$$

It is 2-plectic (Baez, Hoffnung, Rogers) if $c_m \in \Lambda^3 T^*_m M$ is non-degenerate $\forall m \in M$

Examples

(1) Extended phase space

$$M = \Lambda^2 T^* N$$

$$c = db$$

b the tautological 2-form

(2) $M = S^1 \times N$ (N, ω) symplectic

$$c = dt \wedge \omega$$

(3) $M = G$ a compact Lie group

$$c(X, Y, Z) = \langle [\theta^L(X), \theta^L(Y)], \theta^L(Z) \rangle$$

$\langle \, , \rangle$ a biinvariant inner product on $\mathfrak{g} = T_e G$

$$\theta^L(X_g) = (L_{g^{-1}})_* X \in \mathfrak{g}$$

$L_g h = gh$, left-multiplication.

2-plectic $\Longleftrightarrow$ G semi-simple

(4) strong KT geometry

(M, I, g) a Hermitian manifold

g – Riemannian metric

I – integrable complex structure

$g(IX, IY) = g(X,Y) \quad \forall X, Y$

$\omega_I(X,Y) := g(IX, Y)$, $\omega_I \in \Omega^2(M)$

This is <u>Kähler</u> if $d\omega_I = 0$

It is <u>strong KT</u> if $\partial_I \bar{\partial}_I \omega_I = 0$

i.e. $dId\omega_I = 0$

$c = -Id\omega_I \in \Omega^3(M)$

is then closed, but can be degenerate.

$\nabla^B = \nabla^{LC} + \frac{1}{2}c$

is the <u>Bismut</u> <u>connection</u>

(5) strong HKT geometry

(M, g, I, J, K) $\qquad IJ = K = -JI$

$c = -Id\omega_I = -Jd\omega_J = -Kd\omega_K$

$\Longrightarrow I, J, K$ integrable

$+ dc = 0$ is <u>strong HKT</u>

E.g. (i) hyperKähler manifolds

(ii) most compact G with $\dim G = 4r$ (Joyce)

<u>Open problem:</u> find other compact simply-connected examples.

Multi-moment maps

Suppose (M, c) is a ~~close~~ strong geometry and G is a <u>connected</u> group of symmetries

i.e. $\quad L_X c = 0 \quad \forall\, \underline{X} \in \mathfrak{g}$

Cartan's formula

$$0 = L_X c = d(X \lrcorner c) + X \lrcorner \cancel{dc}^{\,0}$$

$$= d(X \lrcorner c) \qquad \text{for } \underline{X} \in \mathfrak{g}.$$

For $\underline{Y} \in \mathfrak{g}$,

$$L_Y (X \lrcorner c) = (L_Y X) \lrcorner c = -[X, Y] \lrcorner c$$

$$= d(Y \lrcorner (X \lrcorner c)) + Y \lrcorner \cancel{d(X \lrcorner c)}^{\,0} = d((X \wedge Y) \lrcorner c)$$

Thus for $\underline{p} = \sum \underline{X}_i \wedge \underline{Y}_i \in \Lambda^2 \mathfrak{g}$

$p \lrcorner c \in \Omega^1(M)$ is closed

$$\iff L(\underline{p}) = 0 \quad \text{where } L = [\cdot, \cdot]$$

<u>Definition</u>: the <u>Lie</u> <u>kernel</u> of $\mathfrak{g}$ is

$$\mathcal{P}_{\mathfrak{g}} = \ker\left(L = [\cdot, \cdot] : \Lambda^2 \mathfrak{g} \longrightarrow \mathfrak{g}\right)$$

Examples: $\mathfrak{g}$ Abelian $\Rightarrow \mathcal{P}_{\mathfrak{g}} = \Lambda^2 \mathfrak{g}$

$\mathfrak{g}$ semi-simple $\Rightarrow \Lambda^2 \mathfrak{g} = \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}$

and (hard) $\mathcal{P}_{\mathfrak{g}}$ is an irreducible module when $\mathfrak{g}$ is simple

Definition: $\nu : M \longrightarrow \mathcal{P}_{\mathfrak{g}}^*$ is a *multi-moment map* if it is equivariant and

$$d\langle \nu, p\rangle = p \lrcorner c \qquad \forall p \in \mathcal{P}_{\mathfrak{g}}$$

Example: $M = \Lambda^2 T^* N$ $\quad c = db$

$G \leq \mathrm{Diff}(N)$

$$\langle \nu, p \rangle = b(p)$$

Theorem: Suppose $b_1(M) = 0$.
Then ν exists if

either (a) G is compact

or (b) M is compact with a G-invariant volume form

Example: $M = S^1 \times N$ $\quad (N, \omega)$ symplectic

If H acts in a Hamiltonian way N, moment map $\mu : N \longrightarrow \mathfrak{h}^*$

then let

$$G = S^1 \times H$$

$$\mathcal{P}_{\mathfrak{g}} = \mathbb{R} \wedge \mathfrak{h} + \mathcal{P}_{\mathfrak{h}}$$

$$\begin{cases} \langle \nu, p \rangle := 0 & \text{for } p \in \mathcal{P}_{\mathfrak{h}} \\ \langle \nu, T \wedge X \rangle := \langle \mu, X \rangle & \text{for } X \in \mathfrak{h} \end{cases}$$

gives a multi-moment map.

MULTI-MOMENT MAPS

II, Algebra

Recall: • (M,c) strong geometry $c\in\Omega^3(M)$, $dc=0$

• G a connected group of symmetries

Lie kernel $\mathcal{P}_{\mathfrak{g}} = \ker(L=[\cdot\,;\cdot] : \Lambda^2\mathfrak{g} \to \mathfrak{g})$

• a multi-moment map is

$$\nu : M \longrightarrow \mathcal{P}_{\mathfrak{g}}^* \qquad \text{equivariant}$$

with

$$d\langle \nu, p\rangle = p \lrcorner c \qquad \forall p \in \mathcal{P}_{\mathfrak{g}}$$

Today: • algebraic existence & uniqueness
• structure theory
• orbit theory

Have

$$0 \to \mathcal{P}_{\mathfrak{g}} \to \Lambda^2\mathfrak{g} \xrightarrow{L} \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}' \to 0$$

$\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ the derived algebra

so $$\dim \mathcal{P}_{\mathfrak{g}} = \dim \Lambda^2\mathfrak{g} - \dim \mathfrak{g}'$$

Dually

$$0 \to \operatorname{Ann}(\mathfrak{g}') \to \mathfrak{g}^* \xrightarrow{d} \Lambda^2\mathfrak{g}^* \xrightarrow{\pi} \mathcal{P}_{\mathfrak{g}}^* \to 0$$

so $$\mathcal{P}_{\mathfrak{g}}^* \cong \Lambda^2\mathfrak{g}^* / B^1(\mathfrak{g})$$

Theorem:

(a) if $b_2(\mathfrak{g}) = 0$, then ν is unique if it exists

(b) if $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$, then ν exists and is unique

Proof: $\qquad \mathcal{P}_{\mathfrak{g}}^* \cong \Lambda^2\mathfrak{g}^* / B^1(\mathfrak{g})$

$\operatorname{im} d = B^1(\mathfrak{g}) \subset Z^2(\mathfrak{g}) = \ker d$

$$\Longrightarrow \qquad d_{\mathcal{P}} : \mathcal{P}_{\mathfrak{g}}^* \longrightarrow \Lambda^3\mathfrak{g}^*$$

$$d_{\mathcal{P}}(\beta + B^1(\mathfrak{g})) = d\beta \quad \text{well-defined,}$$

G-equivariant

$\ker d_{\mathcal{P}} = \ker d / B^1(\mathfrak{g}) \qquad \operatorname{im} d_{\mathcal{P}} = \operatorname{im} d \subset Z^3(\mathfrak{g})$

(i) $d_{\mathcal{P}}$ injective $\iff b_2(\mathfrak{g}) = 0$

(ii) $\operatorname{im} d_{\mathcal{P}} = Z^3(\mathfrak{g}) \iff b_3(\mathfrak{g}) = 0$

Define

$$\Phi : M \longrightarrow Z^3(\mathfrak{g})$$

$$\langle \Phi(m), \underline{X} \wedge \underline{Y} \wedge \underline{Z} \rangle = c_m(X, Y, Z)$$

When $b_2(\mathfrak{g}) = 0$,

ν is a multi-moment map

$$\iff \qquad d_{\mathcal{P}}\nu = \Phi$$

$\square$

Definition : G (or $\mathfrak{g}$) is (cohomologically) (2, 3)-trivial if
$$b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$$

Rough classification of Lie algebras :

(1) $\mathfrak{g}$ solvable if $\mathfrak{g}^k = 0$ for some $k \geq 1$,
$$\mathfrak{g}^1 = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$$

"upper triangular matrices"

(2) $\mathfrak{g}$ nilpotent if $\mathfrak{g}^{(k)} = 0$ for some $k \geq 1$
$$\mathfrak{g}^{(1)} = \mathfrak{g}', \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}]$$

"strictly upper triangular"

(3) $\mathfrak{g}$ solvable $\Longrightarrow$ $\mathfrak{g}'$ nilpotent

(4) $\mathfrak{g}$ simple if it has no non-trivial proper ideals

(5) $\mathfrak{g}$ semi-simple if it is a direct sum of simple ideals

(6) $\mathrm{rad}(\mathfrak{g})$ is the maximal solvable ideal of $\mathfrak{g}$

(7) $0 \longrightarrow \mathrm{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_{ss} \longrightarrow 0$

$\mathfrak{g}_{ss}$ semi-simple

(8) $\mathfrak{g}$ semi-simple $\Rightarrow$ $b_3(\mathfrak{g}) = \#$ simple summands

Theorem: $\mathfrak{g}$ is $(2,3)$-trivial

$$\Longleftrightarrow \begin{cases} \mathfrak{g} \text{ is solvable} \\ \text{(not a product)} \\ b_1(\mathfrak{g}) = \dim (\mathfrak{g}/\mathfrak{g}') = 1 \\ \& \quad H^i(\mathfrak{g}')^{\mathfrak{g}} = 0 \qquad i = 1,2,3 \end{cases}$$

Example: $\mathfrak{g}$ $(2,3)$-trivial , $\dim \mathfrak{g} = 3$

$$\mathfrak{g} = \mathbb{R}X + \mathfrak{g}'$$

$$\Longrightarrow \quad \mathfrak{g}' = \mathbb{R}^2 \qquad H^1(\mathfrak{g}') = \mathbb{R}^2 = \Lambda^1 \mathfrak{g}^*, \quad H^2(\mathfrak{g}') = \mathbb{R} = \Lambda^2 \mathfrak{g}^*$$

$$H^1(\mathfrak{g}')^X = 0 \quad \Longrightarrow \quad X|_{\mathfrak{g}'} \text{ acts invertibly}$$

$$H^2(\mathfrak{g}')^X = 0 \quad \Longrightarrow \quad \operatorname{Tr} X|_{\mathfrak{g}'} \neq 0$$

$$\Longrightarrow \quad X|_{\mathfrak{g}'} = \begin{cases} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} & \lambda \neq 0 \\ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} & \lambda_i \neq 0, \ \lambda_2 \neq -\lambda_1 \\ \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} & \lambda \neq 0, \ \mu \neq 0 \end{cases}$$

$$\text{Scale } X \longrightarrow \begin{cases} \lambda = 1 \\ \lambda_1 = 1, \quad -1 < \lambda_2 \leq 1 \\ \mu = 1, \ \lambda > 0 \end{cases}$$

$$\mathfrak{g} = \begin{cases} \mathfrak{r}_3 = (0, 21+31, 31) \\ \mathfrak{r}_{3,\lambda_2} = (0, 21, \lambda_2.31) \\ \mathfrak{r}'_{3,\lambda} = (0, \lambda.21+31, -21+\lambda.31) \end{cases}$$

Example: if $\mathfrak{g}'$ has a positive grading

$$\mathfrak{g}' = \mathfrak{k}_1 + \dots + \mathfrak{k}_r$$

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subseteq \mathfrak{k}_{i+j} \qquad \forall i,j$$

Let $X|_{\mathfrak{k}_i}$ be multiplication by i

Then $(\Lambda^k \mathfrak{g}'^*)^X = 0$

so

$$\mathfrak{g} = \mathbb{R}X + \mathfrak{g}' \quad \text{is } (2,3)\text{-trivial}$$

In this case $\quad P_{\mathfrak{g}} \cong \Lambda^2 \mathfrak{g}'$

Remark:

$\mathfrak{k} = (0, 0, 12, 13, 23, 14+25+23,$
$\qquad 16+25+35+24)$

has all derivations nilpotent

so $\mathfrak{k} \neq \mathfrak{g}'$ for any $(2,3)$-trivial $\mathfrak{g}$.

Orbits

If G acts transitively on M then

$$\Phi : M \longrightarrow Z^3(\mathfrak{g})$$

$$\langle \Phi, \underline{X} \wedge \underline{Y} \wedge \underline{Z} \rangle = c(X, Y, Z) \qquad (*)$$

has image $G \cdot \Phi_0 = \mathcal{O}'$

$\mathcal{O}'$ defines a strong geometry on $M = G$ via $(*)$

this descends to $M = G/H$ for any H closed in G with

$$\mathfrak{h} \subset \ker \Phi_0 = \{ \underline{X} \in \mathfrak{g} \mid \underline{X} \lrcorner \Phi_0 = 0 \}$$

Suppose $b_2(\mathfrak{g}) = 0$

then $d_{\mathcal{P}} : \mathcal{P}_{\mathfrak{g}}^* \hookrightarrow Z^3(\mathfrak{g})$

For $\beta_0 \in \mathcal{P}_{\mathfrak{g}}^*$, $\mathcal{O} = G \cdot \beta_0 \xrightarrow{\cong} \mathcal{O}' = G \cdot \Phi_0$

$\Phi_0 = d\beta_0$

$\nu(gH) = g \cdot \beta_0$ is the multi-moment map

<u>Proposition:</u> $\mathcal{O} = G \cdot \beta_0 \subset \mathcal{P}_{\mathfrak{g}}^*$, $b_2(\mathfrak{g}) = 0$, carries a strong geometry in this way only if

$$\operatorname{stab}_{\mathfrak{g}} \beta_0 = \ker(d_{\mathcal{P}} \beta_0)$$

Then $\mathcal{O}$ is 2-plectic and ν is the inclusion $\mathcal{O} \hookrightarrow \mathcal{P}_{\mathfrak{g}}^*$.

Satisfy $\forall X \in T_xM \setminus \{0\}$ $\exists\, p \in \mathcal{P}_{\mathfrak{g}}$ s.t.

$$c(X \wedge p) \neq 0$$

MULTI-MOMENT MAPS

III G_2-geometry

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

$$(p,q)(p',q') = (pp' - \bar{q}'q,\ pq' + \bar{p}'q)$$

$$\overline{(p,q)} = (\bar{p}, -q) \qquad \|v\|^2 = v\bar{v}$$

$$\|vw\|^2 = \|v\|^2\|w\|^2$$

<u>Definition:</u> $G_2 = \operatorname{Aut} \mathbb{O}$

$V = \mathbb{R}^7 = \operatorname{Im} \mathbb{O}$

$\phi_0(x,y,z) = \langle xy, z\rangle, \qquad \phi_0 \in \Lambda^3 V^*$

is fully non-degenerate

$= \langle x \times y, z\rangle$

$$\phi_0 = e_{123} + e_1(e_{45} + e_{67}) + e_2(e_{46} + e_{75}) - e_3(e_{47} + e_{56})$$

<u>determines</u> $g_0 = \sum e_i^2$ and $\mathrm{vol}_0 = e_{1234567}$

via

$$(X \lrcorner \phi_0) \wedge (Y \lrcorner \phi_0) \wedge \phi_0 = 6 g_0(X,Y)\, \mathrm{vol}_0$$

$$G_2 = \mathrm{stab}_{GL(7,\mathbb{R})} \phi_0$$

is compact, 14-dimensional, simple simply-connected and ϕ_0 stable form

G_2 acts transitively on $S^6 \subset \mathbb{R}^7$

$e_1 \lrcorner \phi = e_{23} + e_{45} + e_{67}$

e_1 acts as a complex structure on $e_1^\perp$

$\phi_0|_{e_1^\perp} = \mathrm{Re}(\mathrm{vol}_{\mathbb{C}}) \implies e_1^\perp$ has an $SU(3)$ structure

Definition: $\phi \in \Omega^3(M^7)$ is a <u>G_2-structure</u> if each $\phi_x \in \Lambda^3 T_x^* M$ is linearly equivalent to ϕ_0 on $\mathbb{R}^7$.

$\Rightarrow$ M oriented, with Riemannian g

(M, ϕ) is <u>torsion-free</u> if $d\phi = 0 = d(*\phi)$

$\Rightarrow$ $\mathrm{Hol}(g) \leq G_2$ & g is Ricci-flat

Suppose T^2 acts preserving ϕ
generators U_1, U_2

Multi-moment $\nu : M \longrightarrow \mathbb{R} \cong \mathfrak{g}^*$
is just an invariant function with

$$d\nu = U_1 \wedge U_2 \lrcorner \phi$$
$$= (U_1 \times U_2)^\flat$$

$d\phi = 0 \quad \Rightarrow \quad \omega_1 = U_1 \lrcorner \phi, \quad \omega_2 = U_2 \lrcorner \phi$

$d(*\phi) = 0 \quad \Rightarrow \quad \omega_0 = (U_1 \wedge U_2) \lrcorner *\phi$

closed 2-forms

Proposition:

$$\phi = h^2 \omega_0 \wedge d\nu + \omega_1 \wedge \theta_1 + \omega_2 \wedge \theta_2 + d\nu \wedge \theta_1 \wedge \theta_2$$

where $\theta_i(U_j) = \delta_{ij}$ $\quad \theta_i \in \mathrm{span}\{U_1^\flat, U_2^\flat\}$

and $h^2 = \dfrac{1}{|U_1 \wedge U_2|^2}$

For r a regular value of ν
$\nu^{-1}(r)$ is a hypersurface
and inherits an $SU(3)$-structure

$$\sigma = N \lrcorner \phi = h\, i^*\omega_0 + h^{-1}\theta_1 \wedge \theta_2$$
$$\psi_+ = i^*\phi = i^*\omega_1 \wedge \theta_1 + i^*\omega_2 \wedge \theta_2$$

<u>Proposition</u>: (a) any hypersurface $(Y^6, \sigma, \psi_\pm)$ is half-flat

i.e. $d(\frac{1}{2}\sigma^2) = 0$, $d\psi_+ = 0$

(b) a half-flat $(Y^6, \sigma, \psi_\pm)$ is such a hypersurface if and only if one can solve

$$\begin{cases} \frac{d}{dt}\psi_+ = d\sigma \\ \frac{d}{dt}(\frac{1}{2}\sigma^2) = -d\psi_- \end{cases}$$

near $Y \times 0 \subset Y \times \mathbb{R}$ — the Hitchin flow

The modified flow $\quad h^{*} = \dfrac{1}{|\omega_1 \wedge \omega_2|}$

$$\begin{cases} \frac{d}{dr}\psi_+ = d(h\sigma) \\ \frac{d}{dr}(\frac{1}{2}\sigma^2) = -d(h\psi_-) \end{cases}$$

for $(Y^6, \sigma, \psi_\pm, T^2)$ with analytic data corresponds to (M^7, ϕ, T^2, ν).

Definition: the reduction of (M, ϕ, T^2, ν) at level r is
$$Z = \nu^{-1}(r)/T^2$$

$\dim Z = 4$

$i^*\omega_0, i^*\omega_1, i^*\omega_2$ descend to symplectic $\sigma_0, \sigma_1, \sigma_2$

satisfying
$$\begin{cases} \sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2 \\ h^2 \sigma_0^2 = 2\,\mathrm{vol}_Z = \frac{1}{|U_1|^2}\sigma_1^2 = \frac{1}{|U_2|^2}\sigma_2^2 \\ \sigma_1 \wedge \sigma_2 = 2\langle U_1, U_2\rangle \,\mathrm{vol}_Z . \end{cases}$$

Example: $M = \mathbb{R}^7$ $\quad \phi = \phi_0$ $\quad T^2 \subset G_2$ maximal torus

$\mathbb{R}^7 = \mathbb{R} + \mathbb{C}^3$ $\quad (x, z_1, z_2, z_3) \to (x, e^{i\theta} z_1, e^{i\varphi} z_2, e^{-i(\theta+\varphi)} z_3)$

$$\nu = -\frac{1}{4}\mathrm{Re}(z_1 z_2 z_3)$$

For $r \neq 0$, $\quad Z = \nu^{-1}(r)/T^2 \underset{\text{diffeo}}{\equiv} \mathbb{R}^4$

$$\sigma_0 = \frac{1}{4}(dx \wedge dw + du_2 \wedge du_1)$$
$$\sigma_1 = \frac{1}{2}(dx \wedge du_1 + dw \wedge \eta_2)$$
$$\sigma_2 = \frac{1}{2}(dx \wedge du_2 + \eta_1 \wedge dw)$$

$$\eta_1 = \frac{h^2}{4}(|U_2|^2 du_1 - \langle U_1, U_2\rangle du_2)$$
$$\eta_2 = \frac{h^2}{4}(|U_1|^2 du_2 - \langle U_1, U_2\rangle du_1)$$

not hyperKähler.

Definition: a <u>coherent symplectic triple</u> is $\sigma_0, \sigma_1, \sigma_2$ symplectic forms on Z^4 such that

(i) $\mathrm{Span}\{\sigma_0, \sigma_1, \sigma_2\} \leq \Lambda^2 T^* M$ is a maximal positive subspace

(ii) $\sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2$

Put $Q = (q_{ij})_{i,j=1}^{2}$

$$\sigma_i \wedge \sigma_j = q_{ij}\, \sigma_0^2 \quad , \qquad h = \sqrt{\det Q}$$

Theorem: Let Y be a principle T^2-bundle over Z, connection 1-forms θ_1, θ_2. Then Y carries an $SU(3)$-structure

$$\begin{cases} \sigma = h\sigma_0 + h^{-1}\theta_1 \wedge \theta_2 \\ \psi_+ = \sigma_1 \wedge \theta_1 + \sigma_2 \wedge \theta_2 \end{cases}$$

It is half-flat

$$\iff \begin{cases} \sigma_1 \wedge d\theta_1 + \sigma_2 \wedge d\theta_2 = 0 \\ \sigma_0 \wedge d\theta_1 = 0 = \sigma_0 \wedge d\theta_2 \end{cases}$$

If the data is analytic then it defines a torsion-free G_2-structure with T^2-symmetry on a neighbourhood of

$$Y \times 0 \quad \text{in} \quad Y \times \mathbb{R}$$

Every simply-connected torsion-free G_2-structure with free T^2-symmetry arises in this way.

The flow may be described by the following equations Z:

$$\sigma_0' = 0$$
$$\sigma_1' = -d\theta_2, \quad \sigma_2' = d\theta_1$$

$$g_{11}'\sigma_0^2 = -2\sigma_1 \wedge d\theta_2$$
$$g_{22}'\sigma_0^2 = 2\sigma_2 \wedge d\theta_1$$
$$g_{12}'\sigma_0^2 = \sigma_1 \wedge d\theta_1 + \sigma_2 \wedge d\theta_2$$

$$\sigma_0 \wedge \theta_1' = dg_{12} \wedge \sigma_2 - dg_{22} \wedge \sigma_1$$
$$\sigma_0 \wedge \theta_2' = dg_{11} \wedge \sigma_2 - dg_{12} \wedge \sigma_1$$

Example: Z hyperKähler

$$\sigma_0^2 = \sigma_1^2 = \sigma_2^2, \qquad \sigma_i \wedge \sigma_j = 0 \quad i \neq j$$
$$[\sigma_1], [\sigma_2] \in H^2(\cancel{Z}, \mathbb{Z})$$
$$d\theta_1 = \alpha\sigma_1 + b\sigma_2 \qquad \alpha, a, b \in \mathbb{Z}$$
$$d\theta_2 = a\sigma_1 - \alpha\sigma_2$$

gives half-complete G_2-metrics for $\det \begin{vmatrix} \alpha & b \\ a & -\alpha \end{vmatrix} \geq 0$.

$d\theta_i \notin \operatorname{span}\{\sigma_0, \sigma_1, \sigma_2\}$ is more interesting

Singular behaviour ?

Nearly Kähler, Spin(7), ...