Toric Geometry of G_2-manifolds

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Outline

Previous constructions
- Symplectic geometry
- HyperKähler geometry

G_2 manifolds
- Multi-Hamiltonian actions
- Toric G_2
- Singular behaviour
- Smooth behaviour

The Delzant picture

Compact symplectic toric manifolds	$\longleftrightarrow$	Delzant polytopes

(M^{2n}, ω) symplectic with a *Hamiltonian* action of $G = T^n$: *moment map* G-invariant $\mu\colon M \to \mathfrak{g}^* \cong \mathbb{R}^n$ with

$$d\langle \mu, X\rangle = X \lrcorner\, \omega \qquad \forall\, X \in \mathfrak{g}\,.$$

- $b_1(M) = 0 \implies$ each symplectic T^n-action is Hamiltonian
- $\dim(M/T^n)$ equals dimension of target space of μ
- image is Delzant polytope

$$\mu(M) = \Delta = \{a \in \mathbb{R}^n \mid \langle a, u_k\rangle \leqslant \lambda_k,\ k = 1, \dots, m\}$$

- stabiliser of any point is a (connected) subtorus of dimension $n - \operatorname{rank} d\mu$

HyperKähler manifolds

$(M, g, \omega_I, \omega_J, \omega_K)$ is *hyperKähler* if each $(g, \omega_A = g(A\,\cdot\,, \,\cdot\,))$ is Kähler and $IJ = K = -JI$
Then $\dim M = 4n$ and g is Ricci-flat, holonomy in $Sp(n) \leqslant SU(2n)$

Ricci-flatness implies:

> *if M is compact, then any Killing vector field is parallel so the holonomy of M reduces*

So take (M, g) non-compact and complete instead

Swann (2016) and Dancer and Swann (2017), following Bielawski (1999), Bielawski and Dancer (2000), Goto (1994) and Anderson et al. (1989)

Hypertoric is complete hyperKähler M^{4n} with tri-Hamiltonian $G = T^n$ action: have G-invariant map (*hyperKähler moment map*)

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^* \qquad d\langle \mu_A, X\rangle = X \lrcorner\, \omega_A$$

- $\dim(M/T^n)$ is $3n$, the dimension of target space of μ
- stabiliser of any point is a (connected) subtorus of dimension $n - \frac{1}{3}\operatorname{rank} d\mu$
- Locally (Lindström and Roček, 1983)

$$g = (V^{-1})_{ij}\theta_i\theta_j + V_{ij}(d\mu_I^i d\mu_I^j + d\mu_J^i d\mu_J^j + d\mu_K^i d\mu_K^j),$$

  with (V_{ij}) positive-definite, and harmonic on each $a + \mathbb{R}^3 \otimes v$
- $\mu(M) = \mathbb{R}^{3n}$ with configuration of flats (possibly infinitely many) $H(u_k, \lambda_k) = \{a \in \operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n \mid \langle a, u_k\rangle = \lambda_k\}$
- $n = 1$: $V(p) = c + \sum_{q \in Q \subset \mathbb{R}^3} (2\|p - q\|)^{-1}$, $c \geqslant 0$, $V(p) < +\infty$ at some p

G_2 manifolds

M^7 with $\varphi \in \Omega^3(M)$ pointwise of the form

$$\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{275} - e_{347} - e_{356},$$

$e_{ijk} = e_i \wedge e_j \wedge e_k$

Specifies metric $g = e_1^2 + \cdots + e_7^2$, orientation $\mathrm{vol} = e_{1234567}$ and four-form

$$*\varphi = e_{4567} - e_{2345} - e_{2367} - e_{3146} - e_{3175} - e_{1256} - e_{1247}$$

via

$$6g(X, Y)\,\mathrm{vol} = (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

There is also a cross-product

$$g(X \times Y, Z) = \varphi(X, Y, Z)$$

with $X \times Y \perp X, Y$

Holonomy of g is in G_2 when $d\varphi = 0 = d*\varphi$, a *parallel G_2-structure*

Then g is Ricci-flat

Multi-Hamiltonian actions

Joint work with Thomas Bruun Madsen

(M, α) manifold with closed $\alpha \in \Omega^p(M)$ preserved by $G = T^n$
This is *multi-Hamiltonian* if it there is a G-invariant $\nu\colon M \longrightarrow \Lambda^{p-1}\mathfrak{g}^*$ with

$$d\langle \nu, X_1 \wedge \cdots \wedge X_{p-1}\rangle = \alpha(X_1, \dots, X_{p-1}, \cdot\,)$$

for all $X_i \in \mathfrak{g}$

- take $n > p - 2$
- ν invariant $\iff$ α pulls-back to 0 on each T^n-orbit
- $b_1(M) = 0 \implies$ each T^n-action preserving α is multi-Hamiltonian

For (M, φ) a parallel G_2-structure, can take $\alpha = \varphi$ and/or $\alpha = *\varphi$

Multi-Hamiltonian parallel G_2-manifolds

Proposition

Suppose (M, φ) is a parallel G_2-manifold with T^n-symmetry multi-Hamiltonian for $\alpha = \varphi$ and/or $\alpha = *\varphi$. Then $2 \leqslant n \leqslant 4$.

q: dimension of orbit space M^7/T^n
k: dimension of target of multi-moment map $\Lambda^2\mathbb{R}^n$ and/or $\Lambda^3\mathbb{R}^n$

n	q	α	k	note
2	5	φ	1	Madsen and Swann (2012)
3	4	φ	3	
		$*\varphi$	1	
		φ & $*\varphi$	4	*toric*
4	3	φ	6	Baraglia (2010)
		$*\varphi$	4	

Toric G_2

Definition

A *toric G_2 manifold* is a parallel G_2-structure (M, φ) with an action of T^3 multi-Hamiltonian for both φ and $*\varphi$

Let U_1, U_2, U_3 generate the T^3-action, then $\varphi(U_1, U_2, U_3) = 0$, with multi-moment maps $(\nu, \mu) = (\nu_1, \nu_2, \nu_3, \mu)\colon M \longrightarrow \mathbb{R}^4$

$$d\nu_i = U_j \wedge U_k \lrcorner \varphi = (U_j \times U_k)^\flat \qquad (i\,j\,k) = (1\,2\,3)$$
$$d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi$$

Recall $\varphi = e_{123} - e_{145} - e_{167} - e_{246} - e_{275} - e_{347} - e_{356}$
If U_i are linearly independent at p, then there is a G_2-basis so that $\mathrm{Span}\{U_1, U_2, U_3\} = \mathrm{Span}\{E_5, E_6, E_7\}$. The repeated cross-products of the U_i then generate TM and $(d\nu, d\mu)$ is of full rank 4, so (ν, μ) induces a local diffeomorphism

$$M_0/T^3 \longrightarrow \mathbb{R}^4$$

The flat model

$M = S^1 \times \mathbb{C}^3$
Standard flat $\varphi = \frac{i}{2} dx(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}}) + \mathrm{Re}(dz_{123})$
Preserved by $T^3 = S^1 \times T^2 \leqslant S^1 \times SU(3)$
Stabilisers T^2 at $S^1 \times \{0\}$ and T^1 at $S^1 \times (z_i = 0 = z_j, i \neq j)$
Multi-moment maps

$$4(\nu_1 - i\mu) = z_1 z_2 z_3, \quad 4\nu_2 = |z_2|^2 - |z_3|^2, \quad 4\nu_3 = |z_3|^2 - |z_1|^2$$

Topologically $M/T^3 = \mathbb{C}^3/T^2 = C(S^5)/T^2 = C(S^5/T^2) = C(S^3) = \mathbb{R}^4$
The ring $P(\mathbb{R}^6)^{T^2}$ of invariant polynomials has basis μ, ν_1, ν_2, ν_3 and $t = |z_3|^2$. By Schwarz (1975) any smooth invariant function on $\mathbb{C}^3/T^2$ is a smooth function of these five invariant polynomials. However, they satisfy

$$t(t + 2\nu_2)(t - 2\nu_3) = \nu_1^2 + \mu^2, \quad t \geqslant \max\{0, -2\nu_2, 2\nu_3\} \qquad \text{(S)}$$

The linear projection $(t, \nu, \mu) \mapsto (\nu, \mu)$ is a homeomorphism of this set on to $\mathbb{R}^4$

General picture

Proposition

All isotropy groups of the T^3 action are connected and act on the tangent space as maximal tori in (a) $1 \times SU(3)$, (b) $1_3 \times SU(2)$ *or* (c) 1_7

Local tangent space models are flat model around (a) $S^1 \times (0, 0, 0)$ or (b) $S^1 \times (1, 0, 0)$. (b) is the Hopf fibration, topologically rigid.
At (a) (full), v_2 and v_3 agree with the flat model to order 3, v_1 and μ to order 4. Analysis of the singularity (S) and degree arguments give

Theorem

Let M be a full toric G_2-manifold, then M/T^3 is homeomorphic to a smooth four-manifold. Moreover, the multi-moment map (v, μ) induces a local homeomorphism $M/T^3 \longrightarrow \mathbb{R}^4$.

Configuration data: lines in (μ = constant) of rational slope. Any intersection is triple, with an integrality condition.

Smooth behaviour

$M_0 \to M_0/T^3$ is a principal torus bundle with connection one-forms $\theta_i \in \Omega^1(M_0)$ satisfying $\theta_i(U_j) = \delta_{ij}, \quad \theta_i(X) = 0 \; \forall\, X \perp U_1, U_2, U_3$
On M_0, put

$$B = \big(g(U_i, U_j)\big) \quad \text{and} \quad V = B^{-1} = \frac{1}{\det B}\operatorname{adj} B$$

Theorem

$$g = \frac{1}{\det V}\theta^t \operatorname{adj}(V)\,\theta + dv^t \operatorname{adj}(V)\, dv + \det(V)\, d\mu^2$$

$$\varphi = -\det(V)\, dv_{123} + d\mu dv^t \operatorname{adj}(V)\,\theta + \mathfrak{S}_{i,j,k}\, \theta_{ij} dv_k$$

$$*\varphi = \theta_{123} d\mu + \frac{1}{2\det(V)}\big(dv^t \operatorname{adj}(V)\,\theta\big)^2 + \det(V)\, d\mu\, \mathfrak{S}_{i,j,k}\, \theta_i dv_{jk}$$

Theorem (continued)

Such $(g, \varphi, *\varphi)$ defines a parallel G_2-structure if and only if $V \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ is a positive-definite solution to

$$\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial v_i} = 0 \qquad j = 1, 2, 3 \qquad \text{(divergence-free)}$$

and

$$L(V) + Q(dV) = 0 \qquad \text{(elliptic)}$$

where

$$L = \frac{\partial^2}{\partial \mu^2} + \sum_{i,j} V_{ij} \frac{\partial^2}{\partial v_i \partial v_j}$$

and Q is a quadratic form with constant coefficients

L and Q are preserved up to scale by $GL(3, \mathbb{R})$ change of basis; this specifies Q uniquely

Cf. Chihara (2018)

Proposition

Solutions V to the divergence-free equation are given locally by $A \in C^\infty(M_0/T^3, S^2\mathbb{R}^3)$ via

$$V_{ii} = \frac{\partial^2 A_{jj}}{\partial {v_k}^2} + \frac{\partial^2 A_{kk}}{\partial {v_j}^2} - 2\frac{\partial^2 A_{jk}}{\partial v_j \partial v_k}$$

$$V_{ij} = \frac{\partial^2 A_{ik}}{\partial v_j \partial v_k} + \frac{\partial^2 A_{jk}}{\partial v_i \partial v_k} - \frac{\partial^2 A_{ij}}{\partial {v_k}^2} - \frac{\partial^2 A_{kk}}{\partial v_i \partial v_j}$$

$(i\,j\,k) = (1\,2\,3)$

Diagonal solutions

$V = \mathrm{diag}(V_1, V_2, V_3)$ (divergence-free) and off-diagonal terms in (elliptic)

$$\frac{\partial V_i}{\partial v_i} = 0 \qquad \frac{\partial V_i}{\partial v_j}\frac{\partial V_j}{\partial v_i} = 0 \quad (i \neq j)$$

Either $V = \mathrm{diag}(V_1(v_2,\mu), V_2(v_3,\mu), V_3(v_1,\mu))$ linear in each variable
E.g. $V = \mu 1_3$, $\mu > 0$, full holonomy G_2:

$$g = \frac{1}{\mu}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \mu^2(dv_1^2 + dv_2^2 + dv_3^2) + \mu^3 d\mu^2$$

$$d\theta_i = dv_j \wedge dv_k \quad (ijk) = (123)$$

Or get elliptic hierarchy $V_3 = V_3(\mu)$, $V_2 = V_2(v_3,\mu)$, $V_1 = V_1(v_2, v_3, \mu)$

$$\frac{\partial^2 V_3}{\partial \mu^2} = 0 \quad \frac{\partial^2 V_2}{\partial \mu^2} + V_3\frac{\partial^2 V_2}{\partial {v_3}^2} = 0 \quad \frac{\partial^2 V_1}{\partial \mu^2} + V_2\frac{\partial^2 V_1}{\partial {v_2}^2} + V_3\frac{\partial^2 V_1}{\partial {v_3}^2} = 0$$

E.g. $V_3 = \mu, \quad V_2 = \mu^3 - 3v_3^2, \quad V_1 = 2\mu^5 - 15\mu^2 v_3^2 - 5v_2^2$

Complete examples

The flat model $S^1 \times \mathbb{C}^3$

Bryant and Salamon (1989) metrics and their generalisations by Brandhuber et al. (2001) and Bogoyavlenskaya (2013) on $S^3 \times \mathbb{R}^4$: complete, cohomogeneity one with symmetry group $SU(2) \times SU(2) \times S^1 \times \mathbb{Z}/2 \supset T^3$
— only one-dimensional stabilisers.

References I

- Anderson, M. T., Kronheimer, P. B., and LeBrun, C. (1989), 'Complete Ricci-flat Kähler manifolds of infinite topological type', *Comm. Math. Phys.* 125 (4): 637–42.
- Baraglia, D. (2010), 'Moduli of coassociative submanifolds and semi-flat G_2-manifolds', *J. Geom. Phys.* 60 (12): 1903–18.
- Bielawski, R. (1999), 'Complete hyper-Kähler $4n$-manifolds with a local tri-Hamiltonian $\mathbb{R}^n$-action', *Math. Ann.* 314 (3): 505–28.
- Bielawski, R. and Dancer, A. S. (2000), 'The geometry and topology of toric hyperkähler manifolds', *Comm. Anal. Geom.* 8 (4): 727–60.
- Bogoyavlenskaya, O. A. (2013), 'On a new family of complete Riemannian metrics on $S^3 \times \mathbb{R}^4$ with holonomy group G_2', *Sibirsk. Mat. Zh.* 54 (3): 551–62.

References II

- Brandhuber, A. et al. (2001), 'Gauge Theory at Large N and New G_2 Holonomy Metrics', *Nuclear Phys. B* 611 (1-3): 179–204.
- Bryant, R. L. and Salamon, S. M. (1989), 'On the construction of some complete metrics with exceptional holonomy', *Duke Math. J.* 58: 829–50.
- Chihara, R. (2018), *G_2-metrics arising from non-integrable special Lagrangian fibrations*, Jan., arXiv: 1801.05540 [math.DG].
- Dancer, A. S. and Swann, A. F. (2017), 'Hypertoric manifolds and hyperKähler moment maps', in S. Chiossi et al. (eds.), *Special Metrics and Group Actions in Geometry* (Springer INdAM Series, 23; Springer, Cham), 107–27.
- Goto, R. (1994), 'On hyper-Kähler manifolds of type A_∞', *Geometric and Funct. Anal.* 4: 424–54.

References III

- Lindström, U. and Roček, M. (1983), 'Scalar tensor duality and $N = 1, 2$ nonlinear σ-models', *Nuclear Phys. B* 222 (2): 285–308.
- Madsen, T. B. and Swann, A. F. (2012), 'Multi-moment maps', *Adv. Math.* 229: 2287–309.
- Schwarz, G. W. (1975), 'Smooth functions invariant under the action of a compact Lie group', *Topology*, 14: 63–8.
- Swann, A. F. (2016), 'Twists versus modifications', *Adv. Math.* 303: 611–37.