Algebras for the Partial Map Classifier Monad

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Dedicated to the always helpful Max Kelly

Introduction

The category of algebras for the partial map classifier monad is shown to be the category of posets which have suprema for all subsets with at most one element, and which are *shallow*, by which we mean that for any pair of elements a, b,

$$a \leq b \text{ iff } a \text{ is the supremum of the subset } \{a\} \cap \{b\}.$$

On any elementary topos $\mathcal{E}$, one has the functor which to an object A associates the object $TA = \tilde{A}$ which classifies partial maps into A (cf e.g. [J] 1.2). This functor T carries a monad structure $\mathbf{T}= (T, \eta, \mu)$; it is a submonad of the power "set" monad $\mathbf{P}= (P, \eta, \mu)$, as described in, say, [AL],[Mi], or [J] 5.3. We shall analyze the category of algebras for $\mathbf{T}$, for the category of sets, but our arguments and constructions will be intuitionistically pure, so that everything carries over to an arbitrary elementary topos. In fact, when applied to the "usual" boolean category of sets, our notions are rather trivial, or even laughable; for instance, the category of $\mathbf{T}$-algebras is, in this case, just the category of pointed sets, and the proof of this fact is trivial.

From [Ma] , we know that the category of algebras for the power set monad $\mathbf{P}$ is the category of cocomplete posets, with the structure map being the formation of suprema. Since $TA \subseteq PA$ consists of those subsets of A which have at most one element, it has been conjectured that the category of algebras for T should be some category of posets with the (weak!) cocompleteness property that any subset with at most one element has a supremum, but the question was how to construct an order on A out of an algebra structure $\xi : TA \to A$; for the full power set monad such order is given by:

$$a \leq b \iff b = \xi(\{a, b\}),$$

(which it must be if ξ is to be supremum formation), but for $\mathbf{T}$, this does not work, since $\{a, b\}$ in general has more than one element. The theory of actions by the subobject classifier Ω, developed in Section 2, is going to give us the desired order.

This theory of Ω-actions may have some independent interest. In fact, the main technical theorem is Theorem 2.3, where we derive a shallow ordering out of any "admissible" Ω-action; the result about the category of **T**- algebras is really a corollary, and appears as Theorem 5.3. A final paragraph deals with the question of products of families of weakly cocomplete posets. A preliminary version, containing Theorem 5.3, but Theorem 2.3 only in implicit form, appeared as [K] ; we should warn the readers of [K] that we use the words 'subsingleton' and 'shallow' in a slightly different sense here. I would like to thank Michel Thiebaud, Jørn Schmidt, and Wesley Phoa for stimulating discussions leading to the preliminary version; in particular, Wesley Phoa called my attention to actions of Ω on partial map classifiers, an issue which had been discussed in correspondence between him and Dana Scott in the context of effective domains. And I would like to thank Reinhard Börger and Bill Lawvere for interest and comments that were valuable for the present expanded version.

1 Some order theoretic notions

A set U with at most one element is called a *subterminal* set, because having at most one element is equivalent to the unique map $U \to 1$ being monic. This property may also be described:

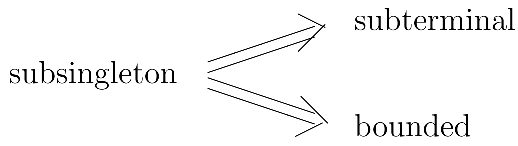
$$\forall x, y \in U : x = y.$$

A subset $U \subseteq A$ of a set A is called a *subsingleton* if there exists some $a \in A$ with $U \subseteq \{a\}$,

$$\exists a \in A\, \forall x \in U : \; x = a.$$

This clearly implies that U is subterminal. (Some sets have the "flabbyness" property that subterminal subsets of it are subsingletons ; but for instance the empty set does not.)

A subset U of a partially ordered set A is called *bounded* if there exists some upper bound for it. Clearly subsingleton subsets of an ordered set are bounded; subterminal subsets need not be (again look at the empty subset of the empty set). A partially ordered set$(A, \leq)$ willl be called *subterminal cocomplete*, respectively *subsingleton cocomplete*, respectively *bounded-cocomplete*) if every subterminal (respectively subsingleton, respectively bounded) subset $U \subseteq A$ has a supremum . Since



the corresponding cocompleteness notions for a poset $(A, \leq)$ relate the opposite way; so that subsingleton cocompletenes (s.s cocompleteness, for short) is the weakest among those considered. The most important for the present purpose is the subterminal-cocompleteness, for which we also use the word *weak* cocompleteness. An order preserving map between s.s. cocomplete posets will be called s.s. cocontinuous if it preserves suprema of s.s. subsets, and similarly for the other notions. Note that the empty set is s.s. cocomplete but not weakly cocomplete. We may remark without proof that s.s. cocompleteness for a poset A is equivalent to A being a *tensored* Ω-enriched category, cf. [Ke] for this notion.

For an s.s. cocomplete poset, the supremum in the following definition is apriori known to exist, but the definition makes sense for any poset:

Definition 1.1 *A poset* $(A, \leq)$ *is* shallow *if for any* $a, b \in A$

$$a \leq b \quad \textit{iff} \quad a = \sup\{x \mid x = b \ \wedge a = b\}.$$

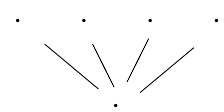
Note that the set over which we are forming supremum here is a subsingleton, and it can be written in various ways:

$$\begin{aligned} &\{x \mid x = b \text{ and } a = b\} \\ &= \{x \mid x = a \text{ and } a = b\} \\ &= \{x \mid x = a \text{ and } x = b\} = \{a\} \cap \{b\}. \end{aligned}$$

Leaving the intuitionistic purity aside for a moment, a weakly cocomplete poset is just a poset with a bottom element $\perp$, namely the supremum of the empty subset, which is the only subterminal subset which is not a singleton. Among the posets with bottom, the shallow ones are characterized by

$$a \leq b \text{ iff } (a = b \text{ or } a = \perp),$$

so they look like



i.e. the bottom is not deep, whence the name "shallow". The word "flat domain" has also been used for such posets, in the boolean case.

2 Order derived from $\Omega-$actions

We consider the set Ω of truth values ($\Omega =$ the subobjectclassifier). It is a monoid under conjunction $\wedge$, with the truth value 'true' as unit. Also, Ω carries a natural partial order, $a \leq b$ iff $a \wedge b = a$.

Definition 2.1 *An action by the monoid Ω on a set A is called* admissible *if for all $a, b \in A$,*

$$[a = b] \cdot a = [a = b] \cdot b, \tag{1}$$

where $[a = b]$ is the truth value of the statement $a = b$.

(Generally, we let the symbol ' [...]' mean 'truth value of'.)

Lemma 2.2 *Given an admissible $\Omega-$action on a set A, and let $a, b \in A$. Then the following three conditions are equivalent*

$$a = [a = b] \cdot a \tag{2}$$

$$a = [a = b] \cdot b \tag{3}$$

$$a = \lambda \cdot b \text{ for some } \lambda \in \Omega. \tag{4}$$

Proof. (2) and (3) are equivalent, by the admissibility assumption. Clearly (3) implies (4). Assume (4). To prove (2) means to prove the first equality sign in (5)

$$\lambda \cdot b = [\lambda \cdot b = b] \cdot \lambda \cdot b = ([\lambda \cdot b = b] \wedge \lambda) \cdot b \tag{5}$$

(the second follows by associativity of the action). But we have $\lambda \leq [\lambda \cdot b = b]$; for, if λ, then $\lambda \cdot b = b$, since $\text{true} \cdot b = b$. So $[\lambda \cdot b = b] \wedge \lambda = \lambda$, and thus the right hand side in (5) equals $\lambda \cdot b$.

Theorem 2.3 *1-168/768 21Given an admissible Ω-action on a set A. For $a, b \in A$, write $a \leq b$ if the equivalent conditions in Lemma 2.2 hold. Then*

1) The binary relation $\leq$ thus defined makes A into a shallow partial order;

2) the action $\Omega \times A \to A$ is order preserving in each variable separately;

3) for each $a \in A$, we have an adjointness

$$\Omega \quad \rightleftharpoons \quad A : \quad - \cdot a \dashv [a \leq -];$$

4) A is conditionally cocomplete, with

$$\sup U \;=\; [b \in U] \cdot b$$

for any bounded set U, and any bound b for it.

5) For each $a \in A$, the poset $\downarrow a$ of elements below a is a frame, isomorphic to a quotient frame of Ω.

(In 2)-5), the notions refer to the order $\leq$ given in 1).)

Proof. We use the criterion (4) in Lemma 2.2 for almost all the arguments. To prove 1): Since $\mathrm{true} \cdot a = a$, $a \leq a$, proving reflexivity. To prove transitivity, assume $a \leq b$ and $b \leq c$, so $a = \lambda \cdot b$ and $b = \mu \cdot c$, for suitable $\lambda, \mu \in \Omega$. So

$$a = \lambda \cdot b = \lambda \cdot (\mu \cdot c) = (\lambda \wedge \mu) \cdot c,$$

so $a \leq c$. Finally, to prove antisymmetry, let $a \leq b$ and $b \leq a$, so $a = \lambda \cdot b$ and $b = \mu \cdot a$. Then

$$\lambda \cdot a = \lambda \cdot (\lambda \cdot b) = (\lambda \wedge \lambda) \cdot b = \lambda \cdot b = a \tag{6}$$

so

$$\begin{aligned} b &= \mu \cdot a = \mu \cdot \lambda \cdot a \\ &= (\mu \wedge \lambda) \cdot a = \lambda \cdot \mu \cdot a = \lambda \cdot b = a, \end{aligned}$$

using associativity of the action, and commutativity of $\wedge$. This proves 1), except for shallowness, which will be proved together with 4) below.
2) If $\lambda \leq \mu$, then $\lambda = \lambda \wedge \mu$, and so

$$\lambda \cdot a = (\lambda \wedge \mu) \cdot a = \lambda \cdot (\mu \cdot a) \leq \mu \cdot a.$$

If $a \leq b$, $a = \mu \cdot b$ for some μ, and so

$$\lambda \cdot a = \lambda \cdot \mu \cdot b = \mu \cdot (\lambda \cdot b) \leq \lambda \cdot b.$$

Before proving the adjointness assertion, we prove

Lemma 2.4 *The partial order $\leq$ on A has binary inf-formation given by*

$$a \wedge b := [a = b] \cdot a = [a = b] \cdot b, \tag{7}$$

(the last equality by the admissibility assumption).

Proof. Clearly $a \wedge b$ thus defined is $\leq a$ and $\leq b$ in virtue of the equations in (7), using (4). Conversely, if $c \leq a$ and $c \leq b$

$$c = [a = c] \cdot c = [b = c] \cdot c$$

by (2), whence

$$\begin{aligned} c &= ([a = c] \wedge [b = c]) \cdot c \\ &= ([a = b] \wedge [a = c] \wedge [b = c]) \cdot c \\ &\leq [a = b] \cdot c \leq [a = b] \cdot b = a \wedge b, \end{aligned}$$

using assertion 2) of the Theorem for the two inequality signs. This proves the Lemma.

3) Clearly $[a \leq -] : A \to \Omega$ is order preserving, and $- \cdot a : \Omega \to A$ is order preserving, by assertion 2) of the Theorem. We have, for $\lambda \in \Omega$,

$$\lambda \leq [x \leq \lambda \cdot x]; \tag{8}$$

for, if λ, $x = \lambda \cdot x$, since $x = \text{true} \cdot x$ (cf. the proof of Lemma 2.2). So if λ, $x \leq \lambda \cdot x$, proving (8). To get the other inequality for adjointness, we use existence of binary infs, as asserted by Lemma 2.4; we have

$$\begin{aligned} [x \leq y] \cdot x &= [(x \wedge y) = x] \cdot x \\ &= [(x \wedge y) = x] \cdot (x \wedge y) \text{ by admissibility} \\ &\leq [(x \wedge y) = x] \cdot y \leq y. \end{aligned}$$

4) Let U be bounded by b. To prove that $[b \in U] \cdot b$ is $\sup(U)$, we first prove that it is an upper bound for U; so let $a \in U$. Then $a = \lambda \cdot b$, since $a \leq b$ by assumption. We should prove $a \leq [b \in U] \cdot b$, or equivalently $\lambda \cdot b \leq [b \in U] \cdot b$. It suffices to prove $\lambda \leq [b \in U]$. But if λ, $b = \lambda \cdot b = a \in U$, so $b \in U$. Conversely, let c be an upper bound for U. To prove $[b \in U] \cdot b \leq c$, it suffices by the adjointness (assertion 4)) to prove $[b \in U] \leq [b \leq c]$. But if $b \in U$, $b \leq c$, since c was asssumed to be a bound for U.

Shallowness of the order is now easy: for any a, b, $\{a\} \cap \{b\}$ is bounded by a, so by 4),

$$\sup(\{a\} \cap \{b\}) = [a \in \{a\} \cap \{b\}] \cdot a = [a = b] \cdot a$$

which equals a iff $a \leq b$, by (2).

To prove the final assertion 5), we first prove that for fixed $a \in A$, the map $- \cdot a : \Omega \to A$ preserves $\wedge$. But

$$\begin{aligned} (\lambda \cdot a) \wedge (\mu \cdot a) &= [\lambda \cdot a = \mu \cdot a] \cdot \lambda \cdot a \\ &= [\lambda \cdot a = \mu \cdot a] \cdot^{\mu} \cdot a \\ &= [\lambda \cdot a = \mu \cdot a] \cdot \lambda \cdot \mu \cdot a, \end{aligned} \tag{9}$$

the last equality since anything of the form $\rho \cdot \lambda \cdot a$ is fixed by multiplication by λ, by the same argument as in (6). From (9) follows

$$(\lambda \cdot a) \wedge (\mu \cdot a) \leq \lambda \cdot \mu \cdot a;$$

the other inequality is obvious, so

$$(\lambda \cdot a) \wedge (\mu \cdot a) = \lambda \cdot \mu \cdot a = (\lambda \wedge \mu) \cdot a$$

Since $\lambda \cdot a \leq a$, it follows that $- \cdot a : \Omega \to A$ factors through $\downarrow a \subseteq A$. As a map $\Omega \to \downarrow a$, it not only preserves $\wedge$ (by the above), but also top element. Also, from assertion 3) in the Theorem, it follows that it has a right adjoint, namely the restriction of $[a \leq -]$ to $\downarrow a$.. Hence it is a frame map; and it is surjective, by the criterion (4) for inequality. So it makes $\downarrow a$ a quotient frame of Ω. (The corresponding nucleus on Ω evidently is the composite of the two adjoints in assertion 3).)

This proves the Theorem.****

Let us apply the Theorem to the case of Ω acting on itself. The derived order is then the natural one. From assertion 4), we then get the following, probably well known,

Corollary 2.5 *Let $U \subseteq \Omega$. Then* $\sup U$ *exists and equals the truth value* [true $\in U$].

3 Admissible Ω-actions derived from order

In the following, more than one partial order on the same set will be considered, prompting us to use the symbol $\sqsubseteq$ for one of these orders.

Theorem 3.1 *Let $A, \sqsubseteq$ be a poset which is subsingleton cocomplete. For $\lambda \in \Omega$ and $a \in A$, we put*

$$\lambda \cdot a := \sup\{x \mid x = a \textit{ and } \lambda\}; \tag{10}$$

this is a unitary and associative action by the monoid $\Omega, \wedge$. It is admissible, and the partial order $\leq$ derived from the action satisfies

$$a \leq b \textit{ implies } a \sqsubseteq b, \tag{11}$$

i.e. is weaker than the original order.

Proof. (All suprema here are with respect to the order $\sqsubseteq$.) We have

$$\text{true} \cdot a = \sup\{x \mid x = a \text{ and true}\} = \sup\{a\} = a,$$

proving that the action is unitary. To prove the associativity assertion,

$$\begin{aligned}
\lambda \cdot (\mu \cdot a) &= \sup\{x \mid x = \mu \cdot a \text{ and } \lambda\} \\
&= \sup\{x \mid x = \sup\{y \mid y = a \text{ and } \mu\} \text{ and } \lambda\} \\
&= \sup\{y \mid y = a \text{ and } \mu \text{ and } \lambda\} \\
&= \sup\{y \mid y = a \text{ and } \lambda \wedge \mu\}
\end{aligned}$$

by the standard rewriting of a sup of sups as a single sup. To prove the admissibility condition

$$[a = b] \cdot a = [a = b] \cdot b, \tag{12}$$

note that the two sides here are formed as suprema of the two sets

$$\{x \mid x = a \text{ and } a = b\}, \text{ resp. } \{x \mid x = b \text{ and } a = b\},$$

but these two sets are equal, by transitive law for equality.

Finally, to prove (11), recall that $a \leq b$ means that for some $\lambda \in \Omega$, $a = \lambda \cdot b$, so

$$a = \lambda \cdot b = \sup\{x = b \text{ and } \lambda\} \sqsubseteq \sup\{x \mid x = b\} = b,$$

the inequality because we are forming supremum over a larger set.

Recall from Section 2 that the order derived from an admissible $\Omega-$action is always conditionally cocomplete. So whether or not $A, \sqsubseteq$ is conditionally cocomplete, it will be conditionally cocomplete in its new (weak) order $\leq$; furthermore

Proposition 3.2 *Let $A, \sqsubseteq$ and $A, \leq$ be as in Theorem 3.1. The identity map on A defines a map*

$$(A, \leq) \longrightarrow (A, \sqsubseteq) \tag{13}$$

(order preserving by (11)), which is conditionally cocontinuous. In particular, the $\Omega-$actions derived from $\leq$ and $\sqsubseteq$ agree.

Proof. Let $U \subseteq A$ be bounded, by $b \in A$, say, w.r.to the order $\leq$. Then by Theorem 2.3, 4),

$$\begin{aligned} \sup_{\leq}(U) &= [b \in U] \cdot b \\ &= \sup_{\sqsubseteq}\{x \mid x = b \text{ and } b \in U\} \\ &\sqsubseteq \sup_{\sqsubseteq}\{x \mid x \in U\}. \\ &= \sup_{\sqsubseteq}(U) \end{aligned}$$

The other inequality is clear, just because we are considering an order preserving map (12). The last assertion follows because the $\lambda \cdot a$'s for the two actions are defined by suprema, for the two orders, of $\{x \mid x = a \text{ and } \lambda\}$, but a set of this form is bounded (by a) in whatever order, so the suprema, for the two orders, of this set agree, by the first assertion of the Proposition.

Proposition 3.3 *Among all orders on A which are subsingleton cocomplete, and which induce a given $\Omega-$action on A, there is a weakest one; and it is shallow, in fact the only shallow one among this class of orders.*

Proof. The weak order $\leq$ induced by the $\Omega-$action is weaker than the others, by (11), and is shallow, by Theorem 2.3. If a shallow order $\sqsubseteq$ is subsingleton cocomplete, and induces the given action,

$$a \sqsubseteq b$$

iff

$$a = \sup\{x \mid x = b \text{ and } a = b\} = [a = b] \cdot b$$

iff

$$a \leq b.$$

4 Admissible Ω-actions derived from T-algebra structures

In this paragraph, $\mathbf{T}= (T, \eta, \mu)$ denotes the partial map classifier monad discussed in the introduction. We have

$$T1 = P1 = \Omega = \text{ set of truth values.}$$

The monoid structure $\wedge$ on this set may be understood in a more sophisticated manner: the monad T is a monoidal monad, in fact a sub-monoidal-monad of the power set monad, whose monoidal structure $\psi : PA \times PB \longrightarrow P(A \times B)$ is described by $\psi(X, Y) = X \times Y \subseteq A \times B$, as in [K1]; the same description yields $\psi : TA \times TB \longrightarrow T(A \times B)$. Since $\mathbf{T}$ is a monoidal monad, $T1$ acquires a monoid structure

$$\psi_{1,1} : T1 \times T1 \longrightarrow T(1 \times 1) \cong T1,$$

which is just the $\wedge$ on Ω. More generally, the monoid $\Omega = T1$ acts on any TA

$$\psi_{1,A} : \Omega \times TA = T1 \times TA \longrightarrow T(1 \times A) \cong TA. \tag{14}$$

The action of $\Omega = T1$ on any TA may be described set theoretically by

$$\lambda \cdot X = \{a \mid a \in X \text{ and } \lambda\}, \tag{15}$$

(for $X \subseteq A$ a subterminal subobject). If $f : A \longrightarrow B$, $Tf : TA \longrightarrow TB$ will preserve the Ω-action.

Now let $\alpha : TA \longrightarrow A$ be a $\mathbf{T}$-algebra structure on A. Then we provide A with an $\Omega-$action by posing, for $\lambda \in \Omega,\ x \in A$

$$\lambda \cdot x := \alpha(\lambda \cdot \{x\}),$$

where $\lambda \cdot \{x\}$ is the action by Ω on TA given by (14) or (15). Let us for the moment call this the *structure induced* action on the algebra (A, α). The question ariseswhether the structure induced action on a (free) algebra (TA, μ_A) equals the action given by (14) or (15). The answer is yes: it amounts to proving that for any subterminal subset $X \subseteq A$,

$$\{x \mid x \in X \text{ and } \lambda\} = \bigcup \{Y \mid Y = X \text{ and } \lambda\}$$

(the right hand side being the one induced by the structure μ_A, since μ is just union formation). If x belongs to the left hand side, $X = \{x\}$ (since X is subterminal), and λ; so $\{x\}$ appears as a Y participating in the union on the right, hence x belongs to the union. Conversely, if x belongs to the union, we have for some subterminal Y that $x \in Y$ and $Y = X$ and λ. So $x \in X$ and λ, proving that x belongs to the left hand side.

It is easy to prove that any $\mathbf{T}$-algebra homomorphism $f : (A, \alpha) \to (B, \beta)$ preserves the structure-induced Ω-actions: for $x \in A$,

$$\begin{aligned} f(\lambda \cdot x) &= f(\alpha(\lambda \cdot \{x\})) = \beta(Tf(\lambda \cdot \{x\})) \\ &= \beta(\lambda \cdot Tf(\{x\})) \text{ since } Tf \text{ preserves action} \\ &= \beta(\lambda \cdot \{f(x)\}) \\ &= \lambda \cdot f(x). \end{aligned}$$

Since $\alpha : TA \to A$ is a $\mathbf{T}$-algebra homomorphism $(TA, \mu_A) \to (A, \alpha)$, it follows that it preserves Ω-action, and since α is a (split) surjection and the Ω-action on TA is an associative and unitary action by the monoid $(\Omega, \wedge)$, it follows that the Ω-action on A is likewise associative and unitary.

Thus we have proved the first two assertions in

Proposition 4.1 *For any $\mathbf{T}$-algebra (A, α), (14) or (15) defines an associative and unitary action by $(\Omega, \wedge)$ on A. Any $\mathbf{T}$-algebra homomorphism commutes with such actions. And such actions are admissible in the sense of Definition 2.1.*

Proof of the last assertion. Let $a, b \in A$. We have

$$\begin{aligned} [a = b] \cdot \{a\} &= \{x \mid x = a \text{ and } a = b\} \\ &= \{x \mid x = b \text{ and } a = b\} \\ &= [a = b] \cdot \{b\}. \end{aligned}$$

Applying α to this equation yields $[a = b] \cdot a = [a = b] \cdot b$, as desired.

With the Ω-action thus defined, the theory developed in Section2 furnishes the underlying set A of any $\mathbf{T}$-algebra (A, α) with a partial order $\leq$, which is conditionally cocomplete and shallow. Explicitly,

$$a \leq b \text{ iff } a = \alpha(\{x \mid x = b \text{ and } \lambda\}) \tag{16}$$

for some $\lambda \in \Omega$. But the order (16) is in fact also weakly cocomplete with α as sup-formation:

Theorem 4.2 *Let (A, α) be a $\mathbf{T}$-algebra. Let $X \subseteq A$ bea subterminal subset with $\alpha(X) = x$. Then $x = \sup X$ with respect to the order $\leq$ described by (16). In particular, $(A, \leq)$ is weakly cocomplete.*

Proof. We first prove that

$$[x \in X] \cdot x = x \tag{17}$$

by proving the following equality of subsets of A

$$[x \in X] \cdot \{x\} = X, \tag{18}$$

from which (17) follows by applying α (which commutes with action). To see (18), let $y \in [x \in X] \cdot \{x\}$. This means that $x \in X$ and $y \in \{x\}$, so $y = x \in X$, so $y \in X$. Conversely let $y \in X$. Then $X = \{y\} = \eta(y)$, (since X is subterminal), so $\alpha(X) = y$, so $x = y$. So $y \in \{x\}$ and $x \in X$. This means $y \in [x \in X] \cdot \{x\}$.

From (17), we can prove the Theorem. Let $y \in X$. Then $X = \{y\}$, whence (as above) $x = y$, whence $y \leq x$, so x is a bound for X. Conversely, suppose z is a bound for X. Then $[x \in X] \leq [x \leq z]$. Since the action is order preserving in the first variable (Theorem 2.3), $[x \in X] \cdot x \leq [x \leq z] \cdot x \leq z$,the last inequality by the back adjunction for the adjointness of Theorem 2.3, 3).

The Theorem is proved.

Proposition 4.3 *Let A be the underlying set of a* $\mathbf{T}$*-algebra. Then A is flabby, in the sense that every subterminal subset of it is a subsingleton.*

Proof. Let X be a subterminal subset; its supremum x exists by the theorem. In the proof of the theorem, we observed that if $y \in X$, then $y = x$; hence $X \subseteq \{x\}$, so X is a subsingleton.

Remark 4.4 There is a rather evident converse of the Theorem. If $(A, \sqsubseteq)$ is a weakly cocomplete poset, supremum formation for subterminals provides a map

$$\sup \; : \; TA \longrightarrow A;$$

this map is a $\mathbf{T}$-algebra structure on A. The order $\leq$ on A induced from the $\mathbf{T}$-algebra structure (via the Ω-action) is weaker than the original order, by Proposition 3.3, since it has the same supremum formation for subterminals, hence the same Ω-action, as the given order $\sqsubseteq$. Also, $\leq$ is shallow, by Theorem 2.3 (1). So if the original order $\sqsubseteq$ is shallow as well, it agrees with $\leq$, by Proposition 3.3.

This and related results will be formulated in functorial terms in the following section.

5 Functorality, and fullness of the functors

We first argue that the construction of Section 2 defines a functor (which preserves underlying sets)

$$\text{Admissible } \Omega\text{-actions} \longrightarrow \text{Subsingleton cocomplete posets;} \tag{19}$$

the morphisms of the two categories here are, respectively, action preserving maps, and order preserving maps which preserve suprema of subsingleton subsets ('s.s. cocontinuous maps'). If $f : A \to B$ is action preserving, it is immediate from the characterization (4) of the constructed order relation $\leq$ that f is order preserving.

Also, if U is a subsingleton subset of A, say $U \subseteq \{a\}$, then U is bounded by a and $f(U)$ by $f(a)$, so by Theorem 2.3 (4),

$$\sup U = [a \in U] \cdot a \text{ and } \sup f(U) = [f(a) \in f(U)] \cdot f(a),$$

so that

$$f \sup U = f([a \in U] \cdot a) = [a \in U] \cdot f(a) \leq [f(a) \in f(U)] \cdot f(a),$$

using $[a \in U] \leq [f(a) \in f(U)]$, so that $f(\sup U) \leq \sup f(U)$. The other inequality is obvious. So f is s.s.cocontinuous, and we have a functor (19).

Proposition 5.1 *The functor (19) is full and faithful.*

Proof. It is faithful because it preserves underlying sets. To see that it is full, let $f : (A, \leq) \to (B, \leq)$ be s.s.cocontinuous, where the orders $\leq$ on A and B are derived from an admissible Ω-action. For $\lambda \in \Omega, a \in A$, we have, by Theorem 2.3 4), applied to the subsingleton $U := \lambda \cdot \{a\} \subseteq \{a\}$:

$$\sup(\lambda \cdot \{a\}) = [a \in \lambda \cdot \{a\}] \cdot a = \lambda \cdot a, \tag{20}$$

so

$$f(\lambda \cdot a) = f(\sup(\lambda \cdot \{a\}) = \sup(f(\lambda \cdot \{a\})),$$

but since $f(\lambda \cdot \{a\}) = \lambda \cdot \{f(a)\}$,

$$\sup(f(\lambda \cdot \{a\}) = \sup(\lambda \cdot \{f(a)\} = \lambda \cdot f(a),$$

the last equality in analogy with (20). This proves that f commutes with action.

The construction of Section 3 defines a functor, likewise preserving underlying sets, in the opposite direction of (19) (and which is actually left inverse of (19)). It is again trivially faithful, but evidently not full: in the category of boolean sets, a map may preserve bottom element without being order preserving.

Next we consider the functorality of the constructions of Section 4. We remarked here that **T** -algebra homomorphisms are also homomorphisms of Ω-actions, so that we have a functor (preserving underlying sets)

$$\mathbf{T} - \textit{algebras} \longrightarrow \Omega\text{-actions}. \tag{21}$$

Proposition 5.2 *The functor (21) is full and faithful.*

Proof. 'Faithful' is again clear. To prove that it is full, let $f : A \to B$ preserve the Ω-action defined by **T**-algebra structures α and β on A and B, cf. (14) or (15). Then by (19) f is s.s.cocontinuous for the order derived from the action. Since subterminals in A and B are subsingletons, by Proposition 3.3, it follows that f is weakly cocontinuous; but by Theorem 4.2, α and β agree with supremum formation for subterminal subsets, so f is an algebra homomorphism.

The results proved so far can be put together in a main theorem:

Theorem 5.3 *The category of* **T***-algebras is equivalent to the full subcategory of the category of weakly cocomplete posets consisiting of those that are furthermore shallow. The equivalence preserves underlying sets.*

Proof. The full and faithful functors of (21) and (19) combine to give a full and faithful

$$\mathbf{T} - \text{algebras} \longrightarrow \text{Admissible } \Omega\text{-actions} \longrightarrow \text{S.s.cocomplete posets.}$$

But its values have underlying sets which are flabby, (Proposition 4.3), and for a flabby set the notions of s.s cocomplete/cocontinuous and weakly cocomplete/cocontinuous evidently agree. So we have a full and faithful functor from **T**-algebras to weakly cocomplete posets. Its values are shallow posets, by Theorem 2.3 (1), since the order comes via an admissible Ω-action. On the other hand, by Remark 4.4, every shallow weakly cocomplete poset arsises this way. This proves the Theorem.

The codomain category in (21) is a topos, so (21) exhibits the category of **T**-algebras as a full subcategory of a topos, raising the question about a pure Ω-action theoretic characterization of this full subcategory. We have intermediate full subcategories

$$\mathbf{T}\text{-algebras} \subseteq \text{Inhabited admissible } \Omega\text{-actions}$$

$$\subseteq \text{Connected } \Omega\text{-actions} \subseteq \Omega\text{-actions} ,$$

where 'connected' in this case means existence of a unique fixpoint for the action (it is clear that if $a \in A$ is an element of an Ω-action, then false$\cdot a$ is a fixpoint for the action; and if a and b are fixpoints for an *admissible* Ω-action, then $a \wedge b = \lambda \cdot a = a$, so $a \leq b$ and similarly $b \leq a$, so fixpoints are unique). These inclusions are all proper.We shall argue this only for the first of them, thereby in particular answer in the negative a question posed by Lawvere (private communication) about the relationship between **T**-algebras and connected Ω-actions. We do this through some general observations.

For any set X, we have a map

$$\Omega \times X \longrightarrow TX \tag{22}$$

which to $\lambda \in \Omega$ and $a \in X$ associates $\{x \mid x = a \text{ and } \lambda\}$. Now TX consists of the subterminal subsets of X, and the image of (5.3) consisis of the subsingletons. For, if $U \subseteq \{a\}$, then

$$U = \{x \mid x \in U\} = \{x \mid x \in U \text{ and } x = a\} = \lambda \cdot \{a\},$$

where $\lambda = [x \in U]$. So (22) is surjective precisely when X is flabby in the sense stated in Corollary 4.3. So if X is not flabby, the image $\Omega \cdot X$ of (22) is a proper sub-Ω-action of TX (containing X), and it will be admissible since TX is. But, being a proper subset of TX (containing X), $\Omega \cdot X$ cannot be a sub **T**-algebra, since TX is generated as a **T**-algebra by X.

If now X is inhabited, the image $\Omega \cdot X$ of (22) will be inhabited, and will thus be an inhabited, admissible, and hence connected Ω-action, but it will not be a **T**-algebra, unless X is flabby. So to argue that an inhabited admissible Ω-action need not be a **T**-algebra, one just has to exhibit a topos and an inhabited object X which is not flabby. Even in the Sierpinski topos Sets^2, it is easy to find such X, say $X = 1 + \frac{1}{2}$ (where $\frac{1}{2}$ is the subobject of 1 intermediate between 0 and 1).

6 On products of cocomplete posets

The following section provides an application of notions from the theory of weakly cocomplete posets to an understanding of products of cocomplete posets $\prod\{A_i \mid i \in I\}$ over an index set which is not necessarily decidable.

It is clear that for any well behaved cocompleteness notion (s.s., weak, conditional, or finitely cocomplete,say; or just cocomplete without qualification), a product $\prod A_i$ of cocomplete posets (with coordinatewise ordering) is again cocomplete, and the projections $\text{proj}_i : \prod A_i \to A_i$ preserve the relevant sup's . Hence they have, in the unqualified case, right adjoints; but they have also left adjoints $\text{in}_i \dashv \text{proj}_i$; these left adjoints exist just under the assumption that the A_i's are weakly (=subterminal-) cocomplete. For the case where I is decidable, these left adjoints are well known, and given by

$$\begin{aligned} \text{proj}_j \text{in}_i(a) &= a \text{ if } i = j \\ &= \bot_j \ \text{ (bottom element of } A_j\text{) if not,} \end{aligned}$$

cf e.g. [JT] p.2 or p.3.

Theorem 6.1 *Let $\{A_i \mid i \in I\}$ be a family of weakly cocomplete posets. Then for each i, $\text{proj}_i : \prod A_i \to A_i$ has a left adjoint in_i (w.r.to the coordinatewise ordering of the product). If $\text{proj}_j \circ \text{in}_i : A_i \to A_j$ is denoted $\delta_{i,j}$, we have*

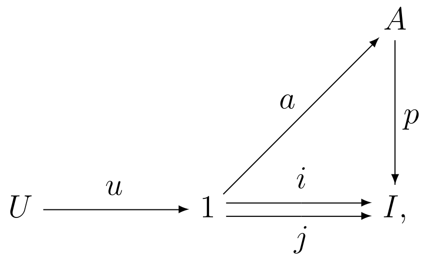
$$\delta_{i,i} = \textit{identity} \tag{23}$$

$$\delta_{j,i}(\delta_{i,j}(a)) = [i = j] \cdot a. \tag{24}$$

$$\delta_{i,j} \textit{ commutes with the action by } \Omega \tag{25}$$

(where the action by the monoid $\Omega = (\Omega, \wedge)$ is constructed as in Section 3).

Proof. The main thing is to construct the $\delta_{i,j}$s. To this end we reformulate the notion of weakly cocomplete into more diagrammatic terms: $(A, \leq)$ is weakly cocomplete iff for every *monic* $u : U \to V$, the order preserving map $\text{hom}(V, A) \to \text{hom}(U, A)$ has a left adjoint lan_u, (and such that furthermore a Beck-Chevalley condition holds). Now consider the family $\{A_i \mid i \in A\}$ of weakly cocomplete posets. For $i, j \in I$, form the equalizer $u : U \to 1$ of (the names of) i and j. For $a \in A_i$, we consider the diagram



where A is the disjoint union of the A_i's (more precisely, to give an I-indexed family of sets A_i *by definition* amounts to give a map $A \to I$). We have $p \circ a = i$ since $a \in A_i$. But then

$$p \circ a \circ u = i \circ u = j \circ u$$

so that $a \circ u$ factors through the inclusion $A_j \to A$. We then define

$$\delta_{i,j}(a) := \mathrm{lan}_u(a \circ u).$$

If $i = j$, u is the identity map, so $\mathrm{lan}_u(a \circ u) = a$, proving (23). To prove (24), we prove two inequalities. To see

$$[i = j] \cdot a \leq \delta_{j,i}(\delta_{i,j}(a))$$

is by the adjointness of Theorem 2.3 equivalent to proving

$$[i = j] \leq [a \leq \delta_{j,i}(\delta_{i,j}(a))].$$

But if $i = j$, we have equality inside the second square bracket, by (23), in particular inequality.

To prove the other inequality, we first give a different description of $\delta_{i,j}(a)$, namely

$$\delta_{i,j}(a) = \bigwedge\{b \in A_j \mid i = j \text{ implies } a \leq b\}; \tag{26}$$

for, $\delta_{i,j}(a)$ itself appears as a b in the intersection, since if $i = j$, $a \leq \delta_{i,j}(a)$ (since $a = \delta_{i,i}(a)$). The other inequality $\leq$ in (26) follows because for each b satisfying $i = j \implies a \leq b$, we have $a \circ u \leq b \circ u$, and so, by the adjointness that defines lan_u, $\mathrm{lan}_u(a \circ u) \leq b$.

So to prove the inequality $\leq$ in (24), we note that by (26)

$$\delta_{j,i}(\delta_{i,j}(a)) = \bigwedge\{b \in A_i \mid j = i \text{ implies } \delta_{i,j}(a) \leq b\}.$$

But inside this infimum participates $b := [i = j] \cdot a$; for, if $j = i$, $\delta_{i,j}(a) = a = [i = j] \cdot a$.

Let now $\mathrm{in}_i : A_i \to \prod A_i$ be given by $\mathrm{proj}_j \circ \mathrm{in}_i; = \delta_{i,j}$, then (23) provides the front adjunction inequality (an equality, actually), whereas for $\underline{a} = (a_i)_{i \in I}$ in $\prod A_i$, it suffices to prove the back adjunction inequality coordinatewise, i.e. to prove

$$\mathrm{proj}_j(\mathrm{in}_i(\mathrm{proj}_i(\underline{a}))) \leq a_j \ \ \forall j \in I.$$

The left hand side here is by (26)

$$\delta_{i,j}(a_i) = \bigwedge \{ b \in A_j \mid i = j \Rightarrow a_i \leq b \},$$

and a_j itself participates in this infimum. This proves the adjointness.

For the last assertion, $\delta_{i,j}$ commutes with Ω-action, since $\delta_{i,j} = \mathrm{proj}_j \circ \mathrm{in}_i$, and both proj_j and in_i preserves supremum formation over subterminals (in_i by being a left adjoint), and these suprema provide the Ω-action. This proves the Theorem.

Remark 6.2 Besides the coordinatewise order on $\prod A_i$, which is usually not shallow, there is also a shallow, weakly cocomplete order on $\prod A_i$, induced by the coordinatewise $\mathbf{T}$-algebra structure. The proj_i preserve the $\mathbf{T}$-induced order as well, since they are $\mathbf{T}$-homomorphisms. It is not difficult to prove that the in_i, constructed as in the proof of the theorem, preserve order, even with the $\mathbf{T}$-induced shallow order on the product. But the back adjunction inequality will not hold in general for the shallow order here.

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