# The dual fibration in elementary terms

Anders Kock

We give an elementary construction of the dual fibration of a fibration. It does not use the non-elementary notion of (pseudo-) functor into the category of categories. In fact, it is clear that the construction we present makes sense for internal categories and fibrations in any exact category.

The dual fibration of a fibration $\mathscr{X} \to \mathscr{B}$ over $\mathscr{B}$ is described in e.g. [Borceux] II.8.3 via a pseudofunctor $F : \mathscr{B}^{op} \to Cat$ (the category of categories), by composing $F$ with the (covarariant!) dualization functor $Cat \to Cat$; choosing such an $F$ is tantamount to choosing a cleavage for the fibration. In the present section, we give an alternative description of the dual fibration, which is elementary and choice-free.

## 1 Fibrations

We recall here some classical notions.

Let $\pi : \mathscr{X} \to \mathscr{B}$ be any functor. For $\alpha : A \to B$ in $\mathscr{B}$, and for objects $X, Y \in \mathscr{X}$ with $\pi(X) = A$ and $\pi(Y) = B$, let $\hom_\alpha(X,Y)$ be the set of arrows $h : X \to Y$ in $\mathscr{X}$ with $\pi(h) = \alpha$. For any arrow $\xi : C \to A$, and any object $Z \in \mathscr{X}$ with $\pi(Z) = C$, post-composition with $h$ defines a map

$$h_* : \hom_\xi(Z,X) \to \hom_{\xi.\alpha}(Z,Y).$$

(we compose from left to right). Recall that $h$ is called *Cartesian* if this map is a bijection, for all such $\xi$ and $Z$.

If $h$ is Cartesian, the injectivity of $h_*$ implies the cancellation property that $h$ is "monic w.r.t. to $\pi$", meaning that for parallel arrows $k, k'$ in $\mathscr{X}$ with codomain $X$, and with $\pi(k) = \pi(k')$, we have that $k.h = k'.h$ implies $k = k'$.

For later use, we recall a basic fact:

**Lemma 1.1** *If $k = k'.h$ is Cartesian, and $h$ is Cartesian then $k'$ is Cartesian.*

The functor $\pi : \mathscr{X} \to \mathscr{B}$ is called a *fibration* if for every $\alpha : A \to B$ in $\mathscr{B}$ and any $Y \in \mathscr{X}$ with $\pi(Y) = B$, there exists a Cartesian arrow over $\alpha$ with codomain $Y$. The *fibre* over $A \in \mathscr{B}$ is the category whose objects are the $X \in \mathscr{X}$ with $\pi(X) = A$, and whose arrows are arrows in $\mathscr{X}$ which by $\pi$ map to $1_A$; such arrows are called *vertical* (over $A$).

All this is standard, dating back essentially to early French category theory (Grothendieck, Chevalley, Giraud, Bénabou,...). For a modern account, see [1] II.8.1, [2] B.1.3, or [3]. Note that these notions are elementary (they make sense for category objects in any left exact category), and they do not depend on the non-elementary notions of *cleavage*, or *Cat*-valued *pseudofunctor*.

## 2 The "factorization system" for a fibration

In the diagrams below, we try to make display vertical arrows vertically, and Cartesian arrows horizontally.

Recall from the literature that if $\pi : \mathscr{X} \to \mathscr{B}$ is a fibration, then every arrow $z$ in $\mathscr{X}$ may be written as a composite of a vertical arrow followed by a cartesian arrow. And, crucially, this decomposition of $z$ is unique modulo a *unique* vertical isomorphism. Or, equivalently, modulo an arrow which is at the same time vertical and cartesian. (Recall that for vertical arrows, cartesian is equivalent to isomorphism (= invertible).) This means that every arrow $z$ in $\mathscr{X}$ may be represented by a pair $(v,h)$ of arrows with $v$ vertical and $h$ cartesian, with $z = v.h$. Thus the codomain of $v$ is the domain of $h$. We call such a pair a "vh composition pair", to make the analogy with vh spans, to be considered below, more explicit. Two such pairs $(v,h)$ and $(v',h')$ represent the same arrow iff there exists a vertical cartesian (necessarily unique, and necessarily invertible) $i$ such that

$$v.i = v' \text{ and } i.h' = h. \tag{1}$$

We say that $(v,h)$ and $(v',h')$ are *equivalent* if this holds. The composition of arrows in $\mathscr{X}$ can be described in terms of representative vh composition pairs, as follows. If $z_j$ is represented by $(v_j,h_j)$ for $j = 1,2$, then $z_1.z_2$ is represented by $(v_1.w, k.h_2)$, where $k$ is cartesian over $\pi(h_1)$ and $w$ is vertical, and the square displayed commutes:

$$\begin{tikzcd}
\cdot \arrow[d, "v_1"'] & & \\
\cdot \arrow[r, "h_1"] \arrow[d, "w"'] & \cdot \arrow[d, "v_2"] & \\
\cdot \arrow[r, "k"'] & \cdot \arrow[r, "h_2"'] & \cdot
\end{tikzcd}$$

Such $k$ and $w$ exists (uniquely, up to unique vertical cartesian arrows): construct first $k$ as a cartesian lift of $\pi(h_1)$, then use the universal property of cartesian arrows to construct $w$.

The arrows $z_1$ and $z_2$ may be inserted, completing the diagram with two commutative triangles, since $z_j = v_j.h_j$. But if we refrain from doing so, we have a blueprint for a succinct and choice-free description of the fibrewise dual $\mathscr{X}^*$ of the fibration $\mathscr{X} \to \mathscr{B}$.

Note that a vh factorization of an arrow in $\mathscr{X}$ is much reminiscent of the factorization for an $E$-$M$ factorization system, as in [Borceux] I.5.5, say, (with the class of vertical arrows playing the role of $E$, and the class of cartesian arrows playing the role of $M$; however, note that not every isomorphism in $\mathscr{X}$ is vertical.

# 3 The dual fibration $\mathscr{X}^*$

The construction presented in this Section is still elementary, but requires more than just left exactness in the category where it is performed, namely exactness; this implies that good quotients exist for equivalence relations, and that maps on such a quotient can be defined by assigning values on representative elements for the equivalence classes. – We present the construction in the exact category of sets, for simplicity.

Given a fibration $\pi : \mathscr{X} \to \mathscr{B}$. We describe another category $\mathscr{X}^*$ over $\mathscr{B}$, as follows: The objects of $\mathscr{X}^*$ are the same as those of $\mathscr{X}$; the arrows $X \to Y$ are represented by vh spans, in the following sense:

**Definition 3.1** *A* vh span *in $\mathscr{X}$ from $X$ to $Y$ is a diagram in $\mathscr{X}$ of the form*

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "v"'] & Y \\ X & \end{tikzcd} \tag{2}$$

*with $v$ vertical and $h$ cartesian.*

The set of arrows in $\mathscr{X}^*$ from $X$ to $Y$ are equivalence classes of vh spans from $X$ to $Y$, for the equivalence relation $\equiv$ given by $(v,h) \equiv (v',h')$ if there exists a vertical isomorphism $i$ (necessarily unique) in $\mathscr{X}$ so that

$$i.v. = v' \text{ and } i.h = h'. \tag{3}$$

We denote the equivalence class of the vh span $(v,h)$ by $\{(v,h)\}$. They are the arrows of $\mathscr{X}^*$; the direction of a the arrow $\{(v,h)\}$ is determined by its cartesian part $h$.

Composition has to be described in terms of representative pairs; it is in fact the standard composite of spans, but let us be explicit: If $z_j$ is represented by $(v_j,h_j)$ for $j = 1,2$, then $z_1.z_2$ is represented by $(w,k)$, where $k$ is cartesian over $\pi(h_1)$ and $w$ is vertical, and the square displayed commutes:

$$\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "w"'] & \cdot \arrow[r, "h_2"] \arrow[d, "v_2"] & \cdot \\ \cdot \arrow[r, "h_1"'] \arrow[d, "v_1"'] & \cdot & \\ \cdot & & \end{tikzcd} \tag{4}$$

Such $k$ and $w$ exists (uniquely, up to unique vertical cartesian arrows): construct first $k$ as a cartesian lift of $\pi(h_1)$, then use the universal property of cartesian arrows to construct $w$. (The square displayed will then actually be a pull-back diagram, thus the composition described will be the standard composition of spans.)

Composition of vh spans does not give a definite vh span, but rather an equivalence class of vh spans. So referring to (4), the composite of $\{(v_1,h_1)\}$ with the of $\{(v_2,h_2)\}$ is defined by

$$\{(v_1,h_1)\}.\{(v_2,h_2)\} := \{(w.v_1,k.h_2)\}.$$

There is a functor $\pi^*$ from $\mathscr{X}^*$ to $\mathscr{B}$; on objects, it agrees with $\pi : \mathscr{X} \to \mathscr{B}$; and $\pi^*(\{(v,h)\}) = \pi(h)$. Note that if $v : X' \to X$ is vertical, the vh span $(v,1)$ represents a morphism $X \to X'$ in $\mathscr{X}^*$.

Clearly, a vertical arrow in $\mathscr{X}^*$ has a unique representative span of the form $(v,1)$. So the fibres of $\pi^* : \mathscr{X}^* \to \mathscr{B}$ are canonically isomorphic to the duals of the fibres of $\pi : \mathscr{X} \to \mathscr{B}$, i.e. $(\mathscr{X}^*)_A \cong (\mathscr{X}_A)^{op}$; so $\mathscr{X}^*$ is "fibrewise dual" to $\mathscr{X}$ (but is not in general dual to $\mathscr{X}$, since the functor $\pi^* : \mathscr{X}^* \to \mathscr{B}$ is still a covariant functor). The arrows in $\mathscr{X}^*$, we call *comorphisms*; it is ususally harmless to use the name "comorphism" also for a representing vh span $(v,h)$.

There are two special classes of comorphisms: the first class consists of those comorphisms that can be represented by a pair $(v,1)$ where 1 is the relevant identity arrow. They are precisely the vertical arrows for $\mathscr{X}^* \to \mathscr{B}$. – The second class consists of those comorphisms that can be represented by a pair $(1,h)$ where 1 is the relevant identity arrow. We shall see that these are precisely the cartesian morphisms in $\mathscr{X}^*$. ’

We first note that if $(v,h)$ represents an arbitrary arrow in $\mathscr{X}^*$, then

$$(v,h) \in \{(v,1)\}.\{(1,h)\}; \tag{5}$$

this is witnessed by the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[r, "h"] \arrow[d, "1"] & \cdot \\
\cdot \arrow[r, "1"'] \arrow[d, "v"'] & \cdot & \\
\cdot & &
\end{tikzcd}$$

since the upper left square is of the form considered in (4).

**Proposition 3.2** *An arrow $g$ is cartesian in $\mathscr{X}^*$ iff it admits a vh representative of the form $(1,h)$.*

**Proof.** In one direction, let $(1,h)$ represent a comorphism $Y \to Z$ over $\beta \in \mathscr{B}$, and let $(v,k)$ represent a comorphism $X \to Z$ over $\alpha.\beta$. We display these data as the full arrows in the following display (in $\mathscr{X}$ and $\mathscr{B}$):

$$\begin{tikzcd}
\cdot \arrow[d, "v"'] \arrow[dr, dotted, "k'"] \arrow[drr, "k"] & & \\
X & Y \arrow[r, "h"'] & Z; \\
\vdots & \vdots & \vdots \\
\cdot \arrow[r, "\alpha"'] & \cdot \arrow[r, "\beta"'] & \cdot
\end{tikzcd}$$

The dotted arrow $k'$ comes about by using the universal property of the cartesian arrow $h$ in $\mathscr{X}$. Since $k$ and $h$ are Cartesian, then so is $k'$, by the Lemma 1.1. So $(v,k')$ is a comorphism over $\alpha$, and $(v,k').(1.h) \equiv (v,k)$, and using the cancellation property of Cartesian arrows, $(v,k')$ is easily seen to be the unique comorphism over $\alpha.\beta$ composing with $(1,h)$ to give $(v,k)$.

In the other direction, let $g$ be a cartesian arrow in $\mathscr{X}^*$. Let $(w,k)$ be an arbitrary representative of $g$. Then by (5), $g = \{(w,1)\}.\{(1,k)\}$. Since $g$ is assumed cartesian in $\mathscr{X}^*$, and $\{(1,k)\}$ is cartesian by what is already proved, it follows from Lemma 1.1 that $\{(w,1)\}$ is cartesian. Since it is also vertical, it follows that it is an isomorphism in $\mathscr{X}^*$, hence $w$ is an isomorphism in $\mathscr{X}$. Since $k$ is cartesian in $\mathscr{X}$, $w^{-1}.k$ is cartesian as well, and

$$(w,k) \equiv (1, w^{-1}.k),$$

so $g$ has a representative of the claimed form.

**Proposition 3.3** *The functor $\pi^* : \mathscr{X}^* \to \mathscr{B}$ is a fibration over $\mathscr{B}$*

**Proof.** Let $\beta : A \to B$ be an arrow in $\mathscr{B}$, and let $Y$ be an object in over $B$. Since $\mathscr{X} \to \mathscr{B}$ is a fibration, there exists in $\mathscr{X}$ a cartesian arrow $h$ over $\beta$, and then the vh span $(1,h)$ represents, by the above, a cartesian arrow in $\mathscr{X}^*$ over $\beta$.

Since $\mathscr{X}^* \to \mathscr{B}$ is a fibration, we may ask for its fibrewise dual $\mathscr{X}^{**}$:

**Proposition 3.4** *There is a canonical isomorphism over $\mathscr{B}$ between $\mathscr{X}$ and $\mathscr{X}^{**}$.*

**Proof.** We describe an explicit functor $y : \mathscr{X} \to \mathscr{X}^{**}$. Let us denote arrows in $\mathscr{X}^*$ by dotted arrows; they may be presented by vh spans $(v,h)$ in $\mathscr{X}$. We first describe $y$ on vertical and cartesian arrows separately. For a vertical $v$ in $\mathscr{X}$, say $v : X' \to X$, we have the vh span $(v,1)$ in $\mathscr{X}$, which represents a vertical arrow $\overline{v} : X' \dashrightarrow X$ in $\mathscr{X}^*$; thus we have a vh span $(\overline{v},1)$ in $\mathscr{X}^*$, which in turn represents a vertical arrow $X \to X'$ in $\mathscr{X}^{**}$. This arrow, we take as $y(v) \in \mathscr{X}^{**}$. Briefly, $y(v) = ((v,1),1)$. – For a cartesian $h : X' \to Y$ (over $\beta$, say), we have a vh span $(1,h)$ in $\mathscr{X}$, which represents a horizontal arrow $\overline{h} : X' \dashrightarrow Y$ in $\mathscr{X}^*$ (cartesian over $\beta$); thus we have a vh span $(1,\overline{h})$ in $\mathscr{X}^*$, hence an arrow in $\mathscr{X}^{**}$, from $X'$ to $Y$ which we take as $y(h) \in \mathscr{X}^{**}$; briefly, $y(h) = (1,(1,v))$.

Then, for a general $f : X \to Y$ in $\mathscr{X}$, we factor it $v.h$ with $v$ vertical and $h$ cartesian, and put $y(f) := y(v).y(h)$. We leave to the reader to verify that a different choice of $v$ and $h$ gives an equivalent vh span in $\mathscr{X}^*$, thus the same arrow in $\mathscr{X}^{**}$.

Conversely, given an arrow $g : X \to Y$ in $\mathscr{X}^{**}$, represent it by a vh span in $\mathscr{X}^*$, $(\overline{v},\overline{h})$,

$$\begin{tikzcd} X' \arrow[r, dotted, "\overline{h}"] \arrow[d, dotted, "\overline{v}"'] & Y \\ X & \end{tikzcd}$$

Since $\overline{v}$ is vertical, we may pick a representative of $\overline{v}$ in the form $(v,1)$ with $v : X \to X'$, and since $\overline{h}$ is cartesian in $\mathscr{X}^*$, we may pick a representative of it if the form $(1,h)$, with $h : X' \to Y$ in $\mathscr{X}$. Then the composite $v.h : X \to Y$ makes sense in $\mathscr{X}$, and it goes by $y$ to the given $g$.

**Example.** Consider a group homomorphism $\pi : \mathscr{X} \to \mathscr{B}$. It is a fibration iff $\pi$ is surjective. Assume this. Then the fibre (over the unique object $*$ of $\mathscr{B}$) is the kernel $\mathscr{K}$ of $\pi$. Every $h \in \mathscr{X}$ is Cartesian; the vertical arrows are those of $\mathscr{K}$. Then $\mathscr{X}^*$ is canonically isomorphic to $\mathscr{X}$. For, an element (arrow) $(v,h)$

of $\mathscr{X}^*$ may be presented by either $(1, v^{-1}.h)$, so may be presented in the form $(1,x)$. The map $(v,h) \mapsto v^{-1}.h$ gives a canonical isomorphism $J : \mathscr{X}^* \to \mathscr{X}$. This isomorphism preserves $\pi$; note that the $\pi$ for $\mathscr{X}^*$ takes $(v,h)$ to $\pi(h)$. Let us for clarity denote it $\pi'$, so $\pi'\{(v,h)\} = \pi(h)$. The kernel $\mathscr{K}'$ for $\pi'$ consists of elements which may be represented in the form $(v,1)$ with $v \in \mathscr{K}$, so $\mathscr{K}'$ may, as a set, be identified with $\mathscr{K}$ by identifying $(v,1) \in \mathscr{K}' \subseteq \mathscr{X}^*$ with $v \in \mathscr{K} \subseteq \mathscr{X}$. But this identification is an anti-isomorphism, since $(v,1)$ by $J$ goes to $v^{-1}.1 = v^{-1}$. So $\mathscr{K}'$ is identified as a group with $\mathscr{K}^{op}$. Thus we have a diagram of group homomorphisms

$$\begin{tikzcd}
\mathscr{K}^{op} \arrow[r, "(-)^{-1}", "\cong"'] \arrow[d, "i"'] & \mathscr{K} \arrow[d, "\subseteq"] \\
\mathscr{X}^* \arrow[r, "J", "\cong"'] \arrow[d, "\pi'"'] & \mathscr{X} \arrow[d, "\pi"] \\
\mathscr{B} \arrow[r, "id"'] & \mathscr{B}
\end{tikzcd}$$

where $i(v) = \{(v,1)\}$. In case where $\mathscr{B} = 1$, and $\mathscr{X}$ is the group $G$, the four maps of the top square are more explicitly‘the four group isomorphisms

$$\begin{tikzcd}
G^{op} \arrow[r, "(-)_i^{-1}"] \arrow[d, "v \mapsto \{(v,1)\}"'] & G \arrow[d, "="] \\
G^* \arrow[r, "J", "\{(v,h)\} \mapsto v^{-1}.h"'] & G
\end{tikzcd}$$

where the inverse of $J$ is given by $h \mapsto \{(1,h)\}$. If we denote the inverse of $J$ by $j$, we can write the information in this diagram more symmetrically:

$$\begin{tikzcd}
G^{op} \arrow[r, "i"] & G^* & G \arrow[l, "j"']
\end{tikzcd}$$

with $i(v) := \{(v,1)\}$ and $j(h) := \{(1,h)\}$.

# 4 The case of a (pseudo-) functor $\mathscr{B}^{op} \to Cat$

It is well known that a pseudofunctor $F : \mathscr{B}^{op} \to Cat$, gives rise to a fibration over $\mathscr{B}$. It is described in, say, [2] B.1.3, or in [1] II.8.3. This fibration is known as the Grothendieck construction for $F$. We descibe it briefly in terms of the factorization system alluded to in Section 1.

Given a functor (or just a pseudo-functor) $F : \mathscr{B}^{op} \to Cat$. Then we have a category $\mathscr{X}$ whose objects are pairs $(X,A)$ with $A$ an object of $\mathscr{B}$ and $X$ an object in $F(A)$. Arrows $(X,A) \to (Y,B)$ are pairs $(v,\alpha)$, where $\alpha : A \to B$ and $v : X \to \alpha^*(Y)$ in $F(A)$ (and where $\alpha^*$ denotes the functor $F(\alpha) : F(B) \to F(A)$). The functor $\pi : \mathscr{X} \to \mathscr{B}$ takes this arrow to $\alpha$.

Let us denote the arrow $(1_{\alpha^*(Y)}, \alpha)$ by $\alpha \triangleleft Y$, thus

$$\alpha^*(Y) \xrightarrow{\alpha \triangleleft Y} Y$$

This is a Cartesian arrow over $\alpha$ in $\mathscr{X}$, and every Cartesian arrow is of this form modulo unique vertical isomorphisms. There is then a canonical factorization of general arrows in $\mathscr{X}$, namely, the arrow given by a pair $(v,\alpha)$, as above, factors as

$$\begin{tikzcd}
(X,A) \arrow[d, "{(v,1_A)}"'] & \\
(\alpha^*(Y),A) \arrow[r, "\alpha \triangleleft Y"'] & (Y,B)
\end{tikzcd}.$$

Let $F'$ be $F$ followed by the dualization functor $Cat \to Cat$. Then a morphism over $\alpha$ in the fibration corresponding to $F'$, from $(X,A)$ to $(Y,B)$, is given similarly, but now with $v : \alpha^*(Y) \to X$, which in terms of the category $F(A)$ rather than $(F(A))^{op}$ may be displayed in terms of the vh span

$$\begin{tikzcd}
(\alpha^*(Y),A) \arrow[r, "\alpha \triangleleft Y"] \arrow[d, "{(v,1_A)}"'] & (Y,B) \\
(X,A) &
\end{tikzcd},$$

and from this, it is clear that the fibration corresponding to $F'$ is isomorphic to $\mathscr{X}^*$ as described in the previous Sections.

One motivation for the present note is to extract the pure category theory behind "fibrewise contravariant functors" (like fibrewise duality for vector bundles), and "star-bundle functors", as in [Kolar et al, 1993] 41.2. This is still an ongoing project.

I cannot imagine that the constructions in the present note are not known, but I do not presently know of any available account.

# References


[1] F. Borceux, *Handbook of Categorical Algebra*, vols. 1-3, Encyclopedia of Mathematics and its Applications 50-52 (Cambridge University Press 1994).

[2] P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, vols. 1-2, Oxford Logic Guides 44 and 45 (Oxford University Press 2002).

[3] A. Kock, Fibrations as Eilenberg-Moore Algebras, arXiv:1312.1608 v1, 2013.

[4] I. Kolar, P.W. Michor, and J. Slovak, *Natural Operations in Differential Geometry*, Springer Verlag 1993.



University of Aarhus, January 2015
mail: kock (at) math.au.dk