Monads on Symmetric Monoidal Closed Categories

By

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Introduction. This note is concerned with "categories with internal hom and $\otimes$-functors", and we shall use the terminology from the paper [2] by Eilenberg and Kelly.

The result proved may be stated briefly as follows: a $\mathscr{V}$-monad ("strong monad") on a symmetric monoidal closed category $\mathscr{V}$ carries two canonical structures as closed functor. If these agree (in which case we call the monad *commutative*), that structure makes the monad into a closed monad, i.e. a monad *in* the hypercategory of closed categories.

In a subsequent note [3] we are going to show how a commutative monad on a symmetric monoidal closed category has for its category of algebras a category which is itself closed in a canonical way (thus extending a theorem of Linton [4]).

1. On strong endofunctors. Throughout, $\mathscr{V}$ denotes a symmetric monoidal closed category in the sense of [2] III.6 (p. 535), i.e.

$$\mathscr{V} = ([\mathscr{V}_0, \otimes, I, r, l, a, c], p, [\mathscr{V}_0, V, \pitchfork, I, i, j, L]).$$

Thus the left hand square bracket ${}^m\mathscr{V}$ is a symmetric monoidal category, and the right hand square bracket ${}^c\mathscr{V}$ is a closed category. We found it convenient to use Zeeman's symbol $\pitchfork$ for the inner hom-functor and write it between its arguments, i.e. $\hom \mathscr{V}(A, B) = A \pitchfork B$. Finally, p is a natural isomorphism

$$(A \otimes B) \pitchfork C \xrightarrow[p_{A,B,C}]{} A \pitchfork (B \pitchfork C). \tag{1.1}$$

By [2], Theorem I.5.2 (p. 445) and Theorem II.6.4 (p. 498), "$\mathscr{V}$ itself" may be considered as a category over ${}^c\mathscr{V}$ as well as over ${}^m\mathscr{V}$. In particular, we have the composition map

$$(B \pitchfork C) \otimes (A \pitchfork B) \xrightarrow{M^B_{AC}} A \pitchfork C. \tag{1.2}$$

Now, a functor $\mathscr{V}_0 \xrightarrow{T} \mathscr{V}_0$ may carry the structure of a closed functor ${}^c\mathscr{V} \to {}^c\mathscr{V}$ (or, equivalently by Proposition II.4.3 (p. 487) in [2], carry the structure of a monoidal functor ${}^m\mathscr{V} \to {}^m\mathscr{V}$). But since $\mathscr{V}$ is a ${}^c\mathscr{V}$-category and a ${}^m\mathscr{V}$-category, T may also carry the structure of a strong functor, that is of a $\mathscr{V}$-functor (in the closed sense, or in the monoidal sense; this again will be equivalent by Theorem II.6.4 (p. 498) in [2]).

Let (T, st^T) be a strong endofunctor on $\mathscr{V}$; so

$$st^T_{A,B}\colon A \pitchfork B \to AT \pitchfork BT,$$

and the axioms VF 1 and VF 2 of [2] (p. 444—445) hold. We can then construct a natural transformation t'' of bifunctors

$$t''_{A,B}\colon A \otimes BT \to (A \otimes B)\,T$$

by letting π^{-1} (the underlying $(pV)^{-1}$ of p^{-1}) act on

$$A \xrightarrow[f_{A,B}]{} B \pitchfork (A \otimes B) \xrightarrow[st^T]{} BT \pitchfork (A \otimes B)\,T,$$

where $f_{A,B}$ is $(1_{A \otimes B})\pi$, i.e. the front adjunction for the adjointness of $-\otimes B$ to $B \pitchfork -$. Using the symmetry c of ${}^m\mathscr{V}$, we may also define

$$t'_{A,B}\colon AT \otimes B \to (A \otimes B)\,T$$

by

$$t'_{A,B} = c \cdot t''_{B,A} \cdot (c)\,T.$$

We shall call $t''_{A,B}$ the canonical right and $t'_{A,B}$ the canonical left transformation associated with (T, st^T), respectively.

Lemma 1.1. *Let $\alpha\colon T \Rightarrow S$ be a strong transformation of strong endofunctors on $\mathscr{V}$, i.e. α satisfies the axiom* VN *of* [2] (p. 466). *Then the diagram*

$$\begin{array}{ccc} A \otimes BT & \xrightarrow{t''_{A,B}} & (A \otimes B)\,T \\ {\scriptstyle 1 \otimes \alpha_B}\downarrow & & \downarrow{\scriptstyle \alpha_{A \otimes B}} \\ A \otimes BS & \xrightarrow{s''_{A,B}} & (A \otimes B)\,S \end{array}$$

commutes, where t'' and s'' are the canonical right transformations associated with T and S respectively. A similar diagram with t', s' also commutes.

Proof. This follows easily from VN and from naturality of $\pi = (p)V$ in the middle variable with respect to α_B.

With notation as in Lemma 1 (except that α is not used), we prove

Lemma 1.2. *The composite strong functor $T \cdot S$ has as its right- and left canonical transformations*

$$A \otimes BTS \xrightarrow{s''_{A,BT}} (A \otimes BT)\,S \xrightarrow{t''_{A,B}S} (A \otimes B)\,TS \tag{1.3}$$

and

$$ATS \otimes B \xrightarrow{s'_{AT,B}} (AT \otimes B)\,S \xrightarrow{t'_{A,B}S} (A \otimes B)\,TS,$$

respectively.

Proof. To prove that (1.3) is the canonical right transformation for TS, it suffices to prove

$$(s''_{A,BT} \cdot t''_{A,B}S)\,\pi = f_{A,B} \cdot st^T \cdot st^S \qquad (= f_{A,B} \cdot st^{TS}). \tag{1.4}$$

In the diagram

$$\begin{array}{ccccc}
A & \xrightarrow{f_{A,BT}} & BT \pitchfork (A \otimes BT) & \xrightarrow{st^S} & BTS \pitchfork (A \otimes BT)S \\
\Big\downarrow {\scriptstyle f_{A,B}} & & \Big\downarrow {\scriptstyle 1 \pitchfork t''_{A,B}} & & \Big\downarrow {\scriptstyle 1 \pitchfork t''_{A,B}S} \\
B \pitchfork (A \otimes B) & \xrightarrow[st^T]{} & BT \pitchfork (A \otimes B)\, T & \xrightarrow[st^S]{} & BTS \pitchfork (A \otimes B)\, TS\,.
\end{array}$$

the right hand square obviously commutes. The left hand square also commutes, for both composites are equal to $(t''_{A,B})\pi$ (by definition and by the fact that f is the front adjunction for π). By definition, the counterclockwise composite of the total diagram is π of the rigth canonical transformation for $T \cdot S$. The clockwise composite is similarly $(s''_{A,BT})\pi \cdot (1 \pitchfork t''_{A,B} S)$ which clearly is $(s''_{A,BT} \cdot t''_{A,B} S)\pi$. But this is (1.4).

The next two lemmas are not concerned with the functor T, but entirely with properties of the adjunction p in (1.1), its front- and end-adjunctions,

$$X \xrightarrow{f_{X,Y}} Y \pitchfork (X \otimes Y)\,, \qquad (X \pitchfork Y) \otimes X \xrightarrow{ev_{X,Y}} Y$$

and the composition M of (1.2). The notation "ev" for the end-adjunction signifies "evaluation". The first lemma then says (in the set-case) that "evaluating a composite map is the same as evaluating twice"; for simplicity, we state it for associative $\otimes$.

Lemma 1.3. *The diagram*

$$\begin{array}{ccc}
(Y \pitchfork Z) \otimes (X \pitchfork Y) \otimes X & \xrightarrow{1 \otimes ev_{X,Y}} & (Y \pitchfork Z) \otimes Y \\
\Big\downarrow {\scriptstyle M^Y_{X,Z} \otimes 1} & & \Big\downarrow {\scriptstyle ev_{Y,Z}} \\
(X \pitchfork Z) \otimes X & \xrightarrow[ev_{X,Z}]{} & Z
\end{array}$$

commutes.

Proof. The lemma is easily derived from the first part of Proposition II.7.3 of [2] (p. 501) together with the associativity of M (as stated in [2], VC 3′ (p. 496)).

The next lemma is derived from Lemma 1.3 by the adjointness π and by the relations the f and the ev therefore have to each other. We omit the details of the proof.

Lemma 1.4. *The diagram*

$$\begin{array}{ccc}
Z \pitchfork (X \otimes Y \otimes Z) & \xleftarrow{M^{Y \otimes Z}_{Z, X \otimes Y \otimes Z}} & ((Y \otimes Z) \pitchfork (X \otimes Y \otimes Z)) \otimes (Z \pitchfork (Y \otimes Z)) \\
\Big\| & & \Big\uparrow {\scriptstyle f_{X,Y \otimes Z} \otimes f_{Y,Z}} \\
Z \pitchfork (X \otimes Y \otimes Z) & \xleftarrow[f_{X \otimes Y, Z}]{} & X \otimes Y
\end{array}$$

commutes.

Both these lemmas are used in proving the following main property of the canonical right transformation t'' associated with a strong endofunctor T as considered above.

Proposition 1.5. *The following diagram commutes:*

$$
(1.5)\qquad
\begin{array}{ccc}
A \otimes (B \otimes CT) & \xleftarrow[\cong]{a} & (A \otimes B) \otimes CT \\
\downarrow{\scriptstyle 1 \otimes t''_{B,C}} & & \\
A \otimes (B \otimes C)\,T & & \downarrow{\scriptstyle t''_{A \otimes B,C}} \\
\downarrow{\scriptstyle t''_{A,B \otimes C}} & & \\
(A \otimes (B \otimes C))\,T & \xleftarrow[aT]{\cong} & ((A \otimes B) \otimes C)\,T\,.
\end{array}
$$

A similar diagram with t' also commutes.

Proof. We start by eliminating the use of the adjointness isomorphism π^{-1} in the definition of t'' by using the well known technique of front- and end-adjunctions. So

$$
(1.6)\qquad t''_{X,Y} = ((f_{X,Y} \cdot st^T) \otimes 1_{YT}) \cdot ev_{YT,(X \otimes Y)T}\,.
$$

Introducing this equation in the diagram (1.5) and leaving out the associativity a from the notation, the counterclockwise composite is

$$
1 \otimes f_{B,C} \otimes 1 \cdot 1 \otimes st \otimes 1 \cdot 1 \otimes ev_{CT,(B \otimes C)T} \cdot f_{A,B \otimes C} \otimes 1 \cdot st \otimes 1 \cdot 1 \otimes ev_{(B \otimes C)T,(A \otimes B \otimes C)T}
$$

which trivially may be written

$$
1 \otimes f_{B,C} \otimes 1 \cdot f_{A,B \otimes C} \otimes 1 \cdot 1 \otimes st \otimes 1 \cdot st \otimes 1 \cdot 1 \otimes ev_{CT,(B \otimes C)T} \cdot ev_{(B \otimes C)T,(A \otimes B \otimes C)T}\,.
$$

This, by Lemma 1.3, is

$$
1 \otimes f_{B,C} \otimes 1 \cdot f_{A,B \otimes C} \otimes 1 \cdot 1 \otimes st \otimes 1 \cdot st \otimes 1 \cdot M^{(B \otimes C)T}_{CT,(A \otimes B \otimes C)T} \otimes 1 \cdot ev_{CT,(A \otimes B \otimes C)T}\,,
$$

and since T is a strong functor, i.e. a $\mathscr{V}$-functor, we may use VF 2′ of [2] (p. 497) to write this as

$$
1 \otimes f_{B,C} \otimes 1 \cdot f_{A,B \otimes C} \otimes 1 \cdot \mathrm{M}^{B \otimes C}_{C,A \otimes B \otimes C} \otimes 1 \cdot st \otimes 1 \cdot ev_{CT,(A \otimes B \otimes C)T}\,.
$$

Finally, we apply Lemma 1.4 and get that this equals

$$
f_{A \otimes B,C} \otimes 1 \cdot st \otimes 1 \cdot ev_{CT,(A \otimes B \otimes C)T}\,,
$$

which by (1.6) is just $t''_{A \otimes B,C}$. This proves commutativity of (1.5). The other half of the proposition is automatic from this, using the coherence of a and c (see e.g. [2], Proposition III.1.1 (p. 512)).

There is also a mixed diagram of the same type as those of the proposition. To prove its commutativity is again almost automatic from (1.5), coherence of a and c and naturality of t''. We state the mixed diagram formally as

Proposition 1.6. *The following diagram commutes*

$$
\begin{array}{ccc}
(A \otimes BT) \otimes C & \xrightarrow{\;a\;} & A \otimes (BT \otimes C) \\
\downarrow{\scriptstyle t''_{A,B} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes t'_{B,C}} \\
(A \otimes B)\,T \otimes C & & A \otimes (B \otimes C)\,T \\
\downarrow{\scriptstyle t'_{A \otimes B,C}} & & \downarrow{\scriptstyle t''_{A,B \otimes C}} \\
((A \otimes B) \otimes C)\,T & \xrightarrow[aT]{} & (A \otimes (B \otimes C))\,T\,.
\end{array}
$$

For the Theorem 1 of the next section we shall need some further lemmas concerned with t'', as well as with the data j of ${}^{c}\mathscr{V}$, l of ${}^{m}\mathscr{V}$, and the adjointness p.

Lemma 1.7. *The diagrams*

$$\begin{array}{ccc} I\otimes X & \xrightarrow{j_X\otimes 1} & (X\pitchfork X)\otimes X \\ & {\scriptstyle l_X}\searrow & \downarrow{\scriptstyle ev_{X,X}} \\ & & X \end{array} \qquad , \qquad \begin{array}{ccc} I & \xrightarrow{j_X} & X\pitchfork X \\ & {\scriptstyle f_{I,X}}\searrow & \uparrow{\scriptstyle 1\pitchfork l_X} \\ & & X\pitchfork(I\otimes X) \end{array}$$

commute.

Proof. This may be rephrased: l_X goes by $\pi = (p)V$ to j_X. This follows easily from the axioms MCC 2, p. 475, (for $A = B = X$) and CC 5, p. 429, of [2].

Lemma 1.8. $t''_{I,A}\cdot l_A T = l_{AT}\colon\ I\otimes AT \to AT.$

Proof. The left hand side is

$$\begin{aligned} f_{I,A}\otimes 1\cdot st\otimes 1\cdot ev_{AT,I\otimes AT}\cdot l_A T &= f_{I,A}\otimes 1\cdot(1\pitchfork l_A)\otimes 1\cdot st\otimes 1\cdot ev_{AT,AT} = \\ &= j_A\otimes 1\cdot st\otimes 1\cdot ev_{AT,AT} = \\ &= j_{AT}\otimes 1\cdot ev_{AT,AT} = \\ &= l_{AT}, \end{aligned}$$

the first equality sign being obvious, the next by Lemma 1.7, the next again by Axiom VF 1 (p. 444) in [2], and the last by Lemma 1.7 again.

2. On strong monads. In this section we prove that the functor part T of a strong monad (T, st), η, μ carries a canonical structure as monoidal (or closed) functor. Precisely, let $\mathscr{V}$ be a symmetric monoidal closed category as in Section 1, let T, η, μ be a monad on $\mathscr{V}_0$ (i.e., $T\colon \mathscr{V}_0 \to \mathscr{V}_0$, $\eta\colon 1_{\mathscr{V}_0}\to T$, $\mu\colon T^2\to T$), and

$$\eta_{AT}\cdot\mu_A = \eta_A T\cdot\mu_A = 1_{AT}, \qquad \mu_{AT}\cdot\mu_A = \mu_A T\cdot\mu_A,$$

let $st_{X,Y}\colon X\pitchfork Y \to XT\pitchfork YT$ make T into a $\mathscr{V}$-functor; then T^2 becomes a $\mathscr{V}$-functor by $st\cdot st$ in an obvious way. The identity functor on $\mathscr{V}_0$ is a $\mathscr{V}$-functor by means of identity maps. With this, it makes sense to require η and μ to satisfy the axiom VN ([2], p. 466); in these cases, VN becomes

$$\text{(2.1)} \qquad \begin{array}{ccc} X\pitchfork Y & \xrightarrow{\ \ 1\ \ } & X\pitchfork Y \\ \downarrow{\scriptstyle st} & & \downarrow{\scriptstyle 1\pitchfork\eta_Y} \\ XT\pitchfork YT & \xrightarrow[\eta_X\pitchfork 1]{} & X\pitchfork YT \end{array},$$

$$\text{(2.2)} \qquad \begin{array}{ccccc} X\pitchfork Y & \xrightarrow{st} & XT\pitchfork YT & \xrightarrow{st} & XT^2\pitchfork YT^2 \\ \downarrow{\scriptstyle st} & & & & \downarrow{\scriptstyle 1\pitchfork\mu_Y} \\ XT\pitchfork YT & & \xrightarrow[\mu_X\pitchfork 1]{} & & XT^2\pitchfork YT. \end{array}$$

We require (2.1) and (2.2) to be satisfied for the monad considered.

Theorem 2.1. *With these data, T becomes a monoidal functor $(T, \psi, \psi^0): m\mathscr{V} \to m\mathscr{V}$ if we put $\psi_{A,B}$ equal to the composite*

$$AT \otimes BT \xrightarrow[t'_{A,BT}]{} (A \otimes BT)\,T \xrightarrow[t''_{A,B}T]{} (A \otimes B)\,T^2 \xrightarrow[\mu_{A\otimes B}]{} (A \otimes B)\,T$$

and by putting $\psi^0 = \eta_I\colon I \to IT$.

Proof. We have to verify the axioms MF 1, MF 2, and MF 3 of [2], p. 473. We shall need

Lemma 2.2. $(\eta_I \otimes 1_A) \cdot t'_{I,A} = \eta_{I\otimes A}$ *and* $(1_A \otimes \eta_I) \cdot t''_{A,I} = \eta_{A\otimes I}$.

Proof. The first assertion will be automatic from the last. The last is straightforward from the strength of η, (2.1).

We proceed to the proof of MF 1, i.e. commutativity of

$$\begin{array}{ccc} IT \otimes AT & \xrightarrow{\psi_{I,A}} & (I \otimes A)\,T \\ {\scriptstyle \eta_I \otimes 1}\uparrow & & \downarrow{\scriptstyle l_A T} \\ I \otimes AT & \xrightarrow[l_{AT}]{} & AT \end{array} \tag{2.3}$$

Replace in this diagram ψ by its defining expression; we then get the composite map to be

$$\eta_I \otimes 1 \cdot t'_{I,AT} \cdot t''_{I,A}\,T \cdot \mu_{I\otimes A} \cdot l_A\,T\,.$$

Using Lemma 2.2 and naturality of η with respect to $t''_{I,A}$, this becomes

$$t''_{I,A} \cdot \eta_{(I\otimes A)T} \cdot \mu_{I\otimes A} \cdot l_A T\,.$$

By the monad equations, this is $t''_{I,A} \cdot l_A\,T$, which by Lemma 1.8 is l_{AT}. This proves MF 1. The proof of MF 2 (which is the "symmetric" of MF 1) is of course much the same: $1 \otimes \eta_I \cdot \psi \cdot r_A\,T$ is

$$1 \otimes \eta_I \cdot t'_{A,IT} \cdot t''_{A,I}\,T \cdot \mu_{A\otimes I} \cdot r_A\,T\,,$$

which is easily seen to equal

$$c \cdot t''_{I,A} \cdot cT \cdot (1_A \otimes \eta_I)\,T \cdot t''_{A,I}\,T \cdot \mu_{A\otimes I} \cdot r_A\,T\,.$$

Using Lemma 2.2, we get

$$c \cdot t''_{I,A} \cdot cT \cdot \eta_{A\otimes 1}\,T \cdot \mu_{A\otimes 1} \cdot r_A\,T\,,$$

which by the monad laws equals

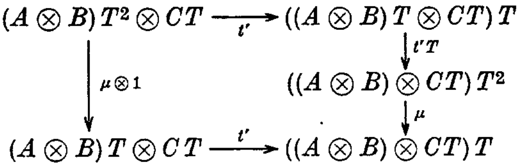
$$c \cdot t''_{I,A} \cdot cT \cdot r_A\,T = c \cdot t''_{I,A} \cdot l_A\,T\,;$$

by Lemma 1.8, again, this is $c \cdot l_{AT}$, that is, r_{AT}.

Finally, we prove MF 3, which in our case says that the diagram (2.4) commutes:

$$\begin{array}{ccc} (AT \otimes BT) \otimes CT & \xrightarrow[a]{\cong} & AT \otimes (BT \otimes CT) \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow{\scriptstyle 1\otimes\psi} \\ (A \otimes B)\,T \otimes CT & & AT \otimes (B \otimes C)\,T \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ ((A \otimes B) \otimes C)\,T & \xrightarrow[aT]{\cong} & (A \otimes (B \otimes C))\,T\,. \end{array} \tag{2.4}$$

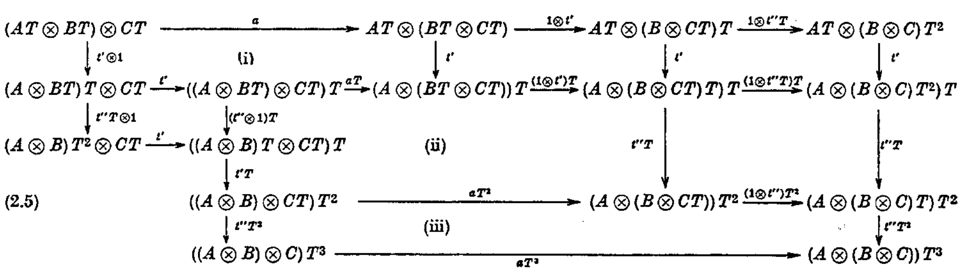
Now, by means of the diagram



which is commutative by Lemma 1.1 and Lemma 1.2, the left hand composite of (2.4) (replacing the ψ's by their defining expressions in terms of t', t'', μ) may be written

$$t'_{A,BT} \otimes 1 \cdot t''_{A,B} T \otimes 1 \cdot t'_{(A \otimes B)T, CT} \cdot t'_{A \otimes B, CT} T \cdot \mu \cdot t''_{A \otimes B, C} T \cdot \mu ,$$

and, by naturality of μ, this again as $\mu_{((A \otimes B) \otimes C)T} \cdot \mu_{(A \otimes B) \otimes C}$ following the left hand composite in the diagram (2.5) below. Similarly, the right hand composite may be written as $\mu_{A \otimes (B \otimes C)} T \cdot \mu_{A \otimes (B \otimes C)}$ following the right hand composite in diagram (2.5). So the monad laws and the commutativity of (2.5) will prove MF 3 and thus the theorem.



(2.5)

But in this, the small uncommented diagrams are naturality squares, whereas (i) and (iii) are commutative by Proposition 1.5 and (ii) by Proposition 1.6. This concludes the proof.

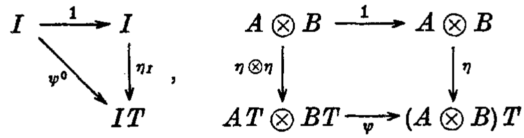
3. On commutative strong monads. The definition of $\psi_{A,B}$ is asymmetric. For reasons of symmetry, we might as well have defined another $\tilde{\psi}_{A,B} \colon AT \otimes BT \to$ $\to (A \otimes B)T$ and proved the theorem for that one. In general, ψ and $\tilde{\psi}$ will be different; so with T, st^T, η, and μ as in the preceding Section, we make the following

Definition 3.1. *The strong monad is called* c o m m u t a t i v e *if ψ and $\tilde{\psi}$ agree, i.e. if*

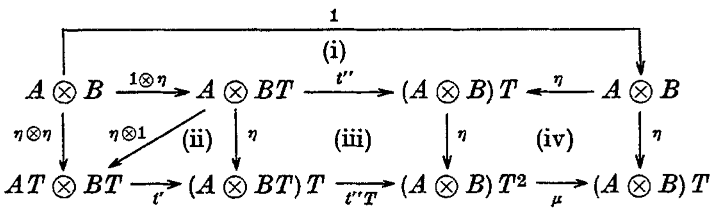
$$t'_{A,BT} \cdot t''_{A,B} T \cdot \mu_{A \otimes B} = t''_{AT,B} \cdot t'_{A,B} T \cdot \mu_{A \otimes B} .$$

Theorem 3.2. *If the monad of Theorem* 2.1 *is commutative, η and μ will be monoidal natural transformations (i.e., satisfy the axioms* MN 1 *and* MN 2 *of* [2], p. 474) *with respect to the monoidal structure ψ, ψ^0 on T, the identity monoidal structure on $1_{\mathscr{V}_0}$, and the monoidal structure derived from ψ, ψ^0 on T^2. Furthermore, T will be symmetric, i.e. satisfy* MF 4 *of* [2], p. 513.

Proof. In this case, MN 1 and MN 2 for η say that the two diagrams

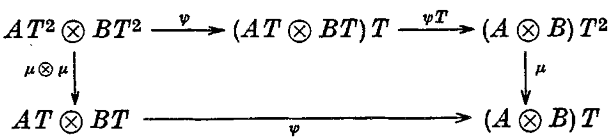


commute. The first is the definition of ψ^0; to see that the second commutes, replace ψ by its defining composite; consider



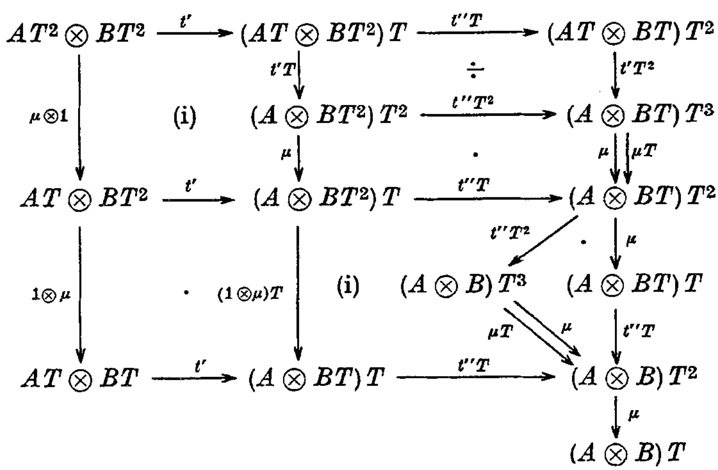
Here (i) and (ii) commute by Lemma 2.2, (iii) by naturality of η, and (iv) by the monad laws. This proves that η is a monoidal transformation. For μ, MN 2 says that the following diagram should commute:

(3.1)



This is the same as the total diagram in the diagram (3.2) below. (Not everything in (3.2) commutes!)

(3.2)



Small diagrams with a dot commute by naturality. The two diagrams with (i) in them commute by Lemma 1.1 and Lemma 1.2 together. In the two places where double arrows occur, the difference is annihilated by the next following arrow, just by the monad laws. Finally, the non-commutative diagram $\div$ is turned into a commutative one when followed by μT, by the assumed commutativity of the monad (Definition 3.1). Then it is easy diagram chasing to conclude that the total diagram of (3.2), and thus (3.1), commutes. This proves MN 2 for μ. MN 1 for μ says $\psi^0 \cdot \psi^0 T \cdot \mu_I = \psi^0$, but since $\psi^0 = \eta_I$, this is just a monad law.

It remains to prove the symmetry of T; MF 4 says in this case that the outer diagram in (3.3) below commutes

$$
(3.3)\qquad
\begin{array}{ccccccc}
AT \otimes BT & \xrightarrow{t'} & (A \otimes BT)\,T & \xrightarrow{t''T} & (A \otimes B)\,T^2 & \xrightarrow{\mu} & (A \otimes B)\,T \\
 & \searrow^{t''} & \div & \nearrow_{t'T} & & & \\
\Big\downarrow{\scriptstyle c} & & (AT \otimes B)\,T & & \Big\downarrow{\scriptstyle cT^2} & & \Big\downarrow{\scriptstyle cT} \\
 & \text{(i)} & \Big\downarrow{\scriptstyle cT} & \text{(i)} & & \text{(ii)} & \\
BT \otimes AT & \xrightarrow[t']{} & (B \otimes AT)\,T & \xrightarrow[t''T]{} & (B \otimes A)\,T^2 & \xrightarrow[\mu]{} & (B \otimes A)\,T\,.
\end{array}
$$

The diagrams (i) commute by definition of t'. The diagram (ii) is naturality of μ. The diagram $\div$ becomes commutative when μ is put on the right, by assumption of commutativity (Definition 3.1) of the monad.

By [2], Proposition II.4.3 (p. 487), we may rephrase (part of) the conclusions of the Theorems 2.1 and 3.2 in terms of ${}^c\mathscr{V}$, so that we have

Corollary 3.4. *The functor part of a strong monad on a symmetric monoidal closed category carries a canonical structure as closed functor; and if the monad is commutative, η and μ are closed transformations.*

4. An application. For this application, it will be convenient to restate the results proved in the language of 2-dimensional categories of EHRESMANN; we stick to the terminology of [2] and call them hyper-categories. In a hyper-category $\mathscr{A}$, the notation "monad in $\mathscr{A}$" makes a sense different from that of "monad on $\mathscr{A}$". A monad *in* $\mathscr{A}$ is a morphism (arrow) in $\mathscr{A}$, $T\colon \boldsymbol{A} \to \boldsymbol{A}$, together with two hyper-morphisms (2-cells) $\eta\colon 1_{\boldsymbol{A}} \to T$ and $\mu\colon T \cdot T \to T$, satisfying the usual identities.

The hyper-category $\mathscr{MCl}$ of monoidal closed categories has as its objects monoidal closed categories, as morphisms monoidal (or closed) functors, as hyper-morphisms functor transformations satisfying MN 1 and MN 2.

By Theorem I.10.7 of [2], p. 469, there is a hyper-functor $*\colon \mathscr{MCl} \to \mathscr{Cat}$ from the hyper-category of monoidal closed categories to the hyper-category $\mathscr{Cat}$ of categories; it assigns to a monoidal closed category $\mathscr{V}$ the category $\mathscr{V}_*$ of $\mathscr{V}$-categories. Now our Theorem 2.1 may be rephrased: a strong monad on a symmetric monoidal closed category carries in a canonical way the structure of a morphism $\mathscr{V} \xrightarrow{T} \mathscr{V}$ in $\mathscr{MCl}$; so $*$ sends it to a morphism

$$T_*\colon \mathscr{V}_* \to \mathscr{V}_*$$

in $\mathscr{Cat}$. It is the one which assigns to a $\mathscr{V}$-category A the $\mathscr{V}$-category with the same set of objects but with new hom-objects $\hom(A, B) = (A \pitchfork B)\,T$ (where $\pitchfork$ is the hom-functor for A). The existence of such a T_* (in the case $\mathscr{V}$ = sets) is probably well-known. I first learned it from LAWVERE.

In the commutative case we get

Theorem 4.1. *Let T, st^T, η, μ be as in Theorem* 3.2. *Then T_* carries the structure of a monad in the hyper-category $\mathscr{S}\mathscr{M}on$ of symmetric monoidal categories (with $\otimes$ in $\mathscr{V}_*$ as defined in* [2], III.3).

Proof. By [2], Proposition III.3.8, $*$ becomes a hyper-functor $\mathscr{S}\mathscr{M}on \to \mathscr{S}\mathscr{M}on$. By our Theorem 3.2, T carries the structure of a monad in the hyper-category $\mathscr{S}\mathscr{M}on$. But a hyper-functor takes monads to monads.

References

[1] J. BÉNABOU, Catégories relatives. C. R. Acad. Sci. Paris **260**, 3824—3827 (1965).

[2] S. EILENBERG and G. M. KELLY, Closed categories. Proc. of the Conference on Categorical Algebra, La Jolla, 1965. Berlin 1966.

[3] A. KOCK, Closed categories generated by commutative monads. J. Austr. Math. Soc. (to appear).

[4] F. E. J. LINTON, Autonomous equational categories. J. Math. Mech. **15**, 637—642 (1966).

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