FWLawvere 24/5 78.

Axioms for Euclidean geom not yet in best form
Maybe not even notion of ring.
Something wrong ~~between~~ with the distinction between synthetic vs. analytic geom

The axioms of Euclid are not so bad.
Problems came when making them rigourous. Nobody would use Hilbert's book for a high school text book.

Math : study of space forms and quantitative relations, and their relationship. Cannot we take this idea directly and arrive at axioms for mathematics. Slightly schematically

$$\begin{tikzcd} S \arrow[loop left, "\text{geom. constr.'s}"] \arrow[r, shift left, "\text{variable quantity}"] & Q \arrow[l, shift left, "\text{coo. syst.}"] \arrow[loop right, "\text{alg. operations}"] \end{tikzcd}$$

space quantity

(Basic variable quantity : distance).
This picture is not good enough to be a cat. Because there are, say, distinct space forms.

Among the objects, we should have the line, L
the plane, P
the space the circle, the sphere

and cartesian products of these things, $L \times L$, $L \times P$, etc.
Eventually we would also have equalizers etc., so that ...

On the other hand we have quantity. But quantity is not just numbers One of them is "Length" $\mathbf{L}$. (confusion of notation), area, volume; these are objects in Q.
There is also pure quantity, and cartesian products of these
E.g.

$$\text{pure quantity} \times \text{length} \longrightarrow \text{length}$$

"multiplication".

A morphism $1 \longrightarrow L$ is a specific length, like this one:

Among maps from 1 to pure quantity, we have

$$1 \xrightarrow{\frac{1}{2}} \text{Pure } Q \qquad 1 \xrightarrow{\pi} \text{Pure } Q.$$

Pure quantity is a commutative ring (perhaps just semiring), is just internal Hom (in the additive sense) from Length to Length, or from area to area.
Length and area are modules

$$\text{Length} \times \text{Length} \xrightarrow{+} \text{Length}$$
$$\text{Length} \times \text{Pure } Q \xrightarrow{\cdot} \text{Length}$$

similarly Area and Volume.

Length and area are line bundles: they are not (canonically) isomorphic to pure quantity;
can even define pure quantity as additive endomorphisms of length.

Very important in Euclidean geometry, have $\neg\Delta$: it is part of Plane × Plane, object whose points (if there are any) are pairs of <u>distinct</u> points (not nec. negation of anything)

For example we can then express "between any two distinct points there is a unique line" by diagrams.

In so called synthetic geom. : "there is only space"
[→ geometric algebra] Or, we may take another one-sided view : there is only ~~pure~~ quantity, and [build arithmetic models].

(From point of view of pure geom, you find eg axioms of Tarski)

What are the main contradictions (axioms, deduction systems) which $C, S \rightleftarrows Q$, satisfy

In synthetic geom., Tarski puts down a quaternary relation 'two pairs of points are equidistant'.
Obscures! We should accept that there is quantity.
We should right out say: there is distance

$$\text{plane} \times \text{plane} \xrightarrow[\text{dist}]{} \text{length}$$

(not into number i.e. pure quantity).
'Metric space' is a confusion, since ~~one~~ here must ask that values are pure numbers, by artificial choice of unit. (A unit of length is a linear isomorphism between pure quantity and length.)
Similarly for space and time.

We should imagine now that we have a cat with finite lim, (and also distributive finite coproducts, perhaps).
No need for 1st order logic, axiom should appear in terms of commutative diagrams, on the space side, which tell you that if you perform this and then that construction you get the same as ...
(E.g. intersection of line and circle)
Using dist, can construct a ~~...~~ circle with given radius and given center.

We all know how to express that pure quantity is a rig object, in terms of commutative diagrams.

A further point, : a basic suggestion: I claim that by this method can simplify axioms in Euclidean geom; the gap between rigourous proof and the actual thinking of it will be less.

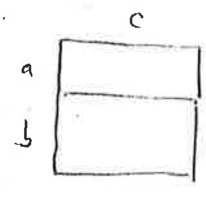
External picture : commutative diagrams ; guide
Internal picture : the geometric picture

So I claim we can have a simplification of logic form of Euclidean geom., but also in actual usability. Also, this .. the external picture may be useful in highschool algebra as well.

E.g. we have our (external) commutative diagram expressing "first take midpoint ; then measure distance ; or first measure distance, then ..

A more striking claim, or conjecture, rather: many of these axioms can be eliminated if we concentrate on the interaction between the two ; just like in topos theory, logic is eliminated. All the Heyting structure follows from the interaction [between $(\ )^{(\ )}$ and Ω]

There are lot of indications of that. For ex., the distributive law in algebra, is not the starting point. We know already that there is something more fundamental than to just postulate the distr. law. In fact the picture



should give the distributive law as a consequence of relationship between quantity and space.

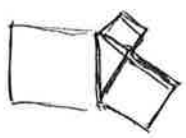
Not yet at all clear, since here we are using area.

In 19th century books I always find area expressed by subdivision etc. Also Moise, Teaching ell. geom "We cannot understand volume."

From our axioms, we will not be able to construct anything bad. So "every subobject of the plane has area" is ok. It is part of the connection between space and quantity.

May have (?) to assume additivity: relation between union and sum.

E.g. the proof of Pythagoras' Thm

 or 

Our understanding of it is based on <u>area</u>. So not rigorous. I don't know the non-area rigorous proof

Probably, we should allow to say that <u>every</u> object has length, area, volume, as well as zero-dim measure. E.g. the zero dimensional measure of the line is ∞, but of a point set

will be 2 (a <u>pure</u> quantity).

...

What does $\mathbb{R}$ (length) classify

$$\begin{tikzcd} ? \arrow[d] \\ X \arrow[r] & \mathbb{R} \text{ length} \end{tikzcd}$$

...

Want to come back to the quantities as such

The reflected structure of ~~space~~ quantity as such is a

Picard closed additive category

(which structure it has in virtue of $S \rightleftarrows Q$.) This means that I have a unit object $\mathbb{R}$ and an object L; as well as $\otimes$ and Hom

(Maybe only semi-additive cat.). – and adjointness

$$\frac{A \otimes B \longrightarrow C}{B \longrightarrow \mathrm{Hom}(B,C)}$$

with $\otimes$ being coherently association and commutative, and R as neutral. (Would be nice to get all these properties as consequence of something)

"Picard" means that the canonical

(1) $$A^* \otimes B \xrightarrow{\text{can.}} \mathrm{Hom}(A,B)$$

(2) $$A^* \otimes A \xrightarrow{\text{can.}} R$$

(when $A^* = \mathrm{Hom}(A,R)$). (The first coming from Cauchy's stress tensor).

(3) $$A \xrightarrow{\text{can}} A^{**}$$

"Picard" means that these 3 are isomorphisms. (2) is the striking ones. (1) and (3) just express a finiteness – ("fin. dim") of A. So (2) is the main axiom. ; "A is invertible." Essentially equivalent (under (1)) to $\mathrm{Hom}(A,A) \underset{\simeq}{\leftarrow} R$.

The claim now is that quantities form a cat like this. A quantity of type A is a map $R \xrightarrow{\alpha} A$. Given $R \xrightarrow{\alpha} A$, $R \xrightarrow{\beta} B$. Form

$$R \underset{\text{can}}{\simeq} R \otimes R \xrightarrow{\alpha \otimes \beta} A \otimes B$$

so we get a quantity of type $A \otimes B$. Call it $\alpha \Box \beta$, say. (Can also compose arrows; in special case 'product' means that 2; if $A = B = R$, you get $r_1 \circ r_2 = r_1 \Box r_2$. Known facts about linear alg / closed cat.).

The invertibility of an obj. says in a crude way that it is 1-dim. In alg. geom., we have the slogan that 'invertible' means "locally ≅ to R"
For an object to be <u>actually</u> isomorphic to R means that it is possible to choose a unit. Locally so should mean that it is possible to locally choose a unit.

A few words about physical quantities
Here we not only have length, area, volume, but also mass, length, time

We want to say area ≅ length ⊗ length.

I don't know any book that systematically considers quadratic forms with values in a line bundle, not R
E.g. measuring <u>area</u>, not ~~...~~ giving numbers.
(The orientation sheaf often comes up, but...). ~~...~~
I claim that the actual mathematical content of high-shool physics ... is things like

energy density ≅ pressure

What I am saying about Picard cat's is a recuperation of the fact that "one cannot add up length and area" Physical quantities do not form a ring but an additive category.
We in fact all learned in high school, but forgot in graduate school.

Could give now a didactic lecture on this .. no time.
Usual notation $L \cdot T^{-1}$ = velocity. This means $\mathrm{Hom}(T, L)$ or $T^* \otimes L$. Now we get a conceptual

content, because a particular velocity is an arrow

$$T \xrightarrow{v} L$$

because velocity transforms time into length

pressure $=$ Hom (area, force)

$\cong$ momentum flux rate ("rate" means T^*, also called frequency). mass^* is called "specific".

Specific volume $=$ $\text{mass}^* \otimes \text{volume}$ $=$ Hom (mass, volume).

In any Picard cat., the objects form a group.

In physics, this is a free group on 3 generators: [[C,G,S]]

(not the same thing as 3-dimensionality of space)

Dimensional analysis.

Period of a pendulum:

'Modelling and scaling'

Claim that the basic content is this kind of closed cat.

Do units exist?

Can light years be comparable with Å?

Very idealistic to say "one chooses globally a unit".

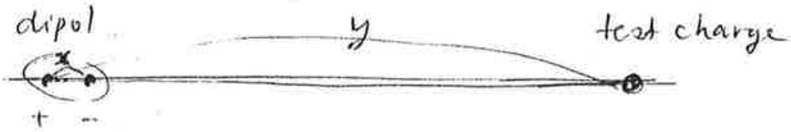
Probably there exist algebraic spaces whose Picard group is free on 3 gen's.

Question raised by Hanne Kock two days ago. She objected to the idea of having $h^2 = 0$ but $h \neq 0$ because it does not seem to be true about length, because its area is there, nonvanishing if the length is nonvanishing.

What we have been calling 'line type' is not a line, but pure quantity.

The point about infinitesimal quantities is that they are ratios of length, say.

There is a physical example, dipol.



It is a Thm of physics that the force excerted by the dipol on test charge is prop. to $\frac{1}{(\text{dist})^3}$ (and is weak) Easy to prove this by pure highschool algebra. Physical way of expressing this is $x << y$. We express it by $x = \lambda y$ with λ a pure quantity with $\lambda^2 = 0$. Then can prove it from inverse square law

$$\frac{1}{(y+x)^2} - \frac{1}{(y-x)^2} = \frac{\lambda}{y^3}$$

Gives rise to a definite math. question:
Do there exist a Picard cat (in a Cartesian cat) in which R has so many nilpotent elements that it is of line type, and yet length × length ⟶ area has trivial kernel. (length × length formed in $\mathcal{E}$).

"Locally iso to R" means w r to a Grothendieck top. e.g. Zariski .. perhaps based on 2 topologies

Question of charge q ; q^2 exists ..
"taking square roots of line bundles (square roots w.r.to exterior powe
Pure speculation.

The question of minus.

There is an antagonistic contradiction between minus and infinity.

I want to permit quantities to be infinite.

For problems of orientation, you may want minus.

I never understood what orientation is.

Need also for Gauss – Green – Stokes Thm.

...

The area dxd problem has to do with problems of distance in line type. Do infinitesimals have length.