FINITENESS OF RELATIVE EQUILIBRIA IN THE PLANAR GENERALIZED N-BODY PROBLEM WITH FIXED SUBCONFIGURATIONS

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Abstract. We prove that a fixed configuration of $N-1$ masses in the plane can be extended to a central configuration of N masses by adding a specified additional mass only in finitely many ways. This holds for a family of potential functions including the Newtonian gravitational case and the classical planar point vortex model.

1. Introduction

In the classical N-body problem, the acceleration due to Newtonian gravity is:

$$\ddot{x}_j = \sum_{i \neq j} \frac{m_i(x_i - x_j)}{r_{ij}^3} \qquad 1 \leq j \leq N \tag{1}$$

where $x_i \in \mathbb{R}^3$ is the position of particle i, r_{ij} is the distance between x_i and x_j, and m_i is the mass of particle i [22].

We will consider this generalized to

$$\ddot{x}_j = \sum_{i \neq j} \frac{m_i(x_i - x_j)}{r_{ij}^D} \qquad 1 \leq j \leq N \tag{2}$$

corresponding to a central potential

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}^{D-2}}$$

Although D can be thought of as a real parameter, in this paper we will only consider integer $D \geq 2$. Newtonian gravitation corresponds to $D = 3$, while systems of planar point vortices have been modelled with $D = 2$. In the vortex case, the potential function is logarithmic, $U = \sum_{i<j} m_i m_j \ln(r_{ij})$[11], and it makes physical sense for the parameters m_i to have negative values as well as positive, corresponding to both directions of rotation.

Our restriction to integer D is due to our reliance on polynomial methods; it would be interesting if a proof could be found for all real $D \geq 2$.

A configuration is called *central* if there exists $\lambda \in \mathbb{R}$ such that

$$\lambda(x_j - c) = \sum_{i \neq j} \frac{m_i(x_i - x_j)}{r_{ij}^D} \qquad 1 \leq j \leq N \tag{3}$$

where c is the center of mass.

Central configurations are of interest for a number of reasons in the N-body problem [18]. In general, they provide landmarks from which other dynamic or global phenomena may be deduced.

While much is known about central configurations there are also many outstanding questions. One of the most famous and interesting problems was first posed by Chazy [5], and then garnered more attention from being posed in a textbook on celestial mechanics by Wintner [27]. More recently, Smale highlighted it as the sixth of his 18 problems for the 21st century [24]. The question is: for any N positive masses are there finitely many planar central configurations up to symmetry?

One of the few results related to Smale's problem which applies to any number of bodies is Forrest Moulton's theorem [21] on collinear central configurations: there is exactly one central configuration for each ordering of N positive masses on a line.

Another result valid for any number of bodies, due to Richard Moeckel, is a generic finiteness result for N bodies in $\mathbb{R}^{N-1}$ [20].

There is also a cluster of work on extending an N-body central configuration to $N+1$ bodies [28, 19, 26], using the implicit function theorem to continue a solution with a zero mass to one or more small masses. Related to this is the result of Lindstrom [16] that an N-body planar configuration (in the Newtonian case) can be extended with a given positive mass in at most finitely many points (assuming the configuration of the original N bodies is fixed).

In this paper we will generalize the result of Lindstrom to a generalized potential and for an arbitrary given additional mass (any real value).

2. Tropical geometry

This section is a brief introduction to tropical geometry which will be needed later. It is convenient to use multi-variable index notation. A polynomial p in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ will be written as $\sum_{v \in \mathcal{A}} a_v x^v$ where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n_{\geq 0}$, $v = (v_1, \ldots, v_n)$, and $a_v \in \mathbb{C} \backslash \{0\}$. The Newton polytope $NP(p)$ of p is the convex hull of $\mathcal{A}$. The initial form $\mathrm{in}_w(p)$ with respect to a *weight* vector $w \in \mathbb{R}^n$ is the sum of the $a_v x^v$ such that $w \cdot v$ is maximal. Geometrically, an initial form is the part of the polynomial lying on a face of its Newton polytope.

The tropical variety $T(p)$ of a single polynomial is the set of vectors $w \in \mathbb{R}^n$ such that the inner product of w with elements of the Newton polytope $NP(p)$ is maximized on at least two vertices of $NP(p)$. In other words, it is the set of vectors w such that $\text{in}_w(p)$ is not a monomial.

Consider a set of polynomials $P = \{p_1, \ldots, p_m\}$ generating an ideal $I = \langle P \rangle$ in the polynomial ring. By the variety of I we mean $V(I) = \{x \in (\mathbb{C} \setminus \{0\})^n | p(x) = 0 \ \forall p \in I\}$. While we always have $V(I) = \bigcap_{p_i \in P} V(\langle p_i \rangle)$ it is not always true that $\bigcap_{p \in I} T(p) = \bigcap_{p_i \in P} T(p_i)$. The first set, $\bigcap_{p \in I} T(p)$, is called the *tropical variety* of the *ideal* I and is denoted $T(I)$ while the second set, $\bigcap_{p_i \in P} T(p_i)$, is the *tropical prevariety* of the *set* P. It is always true that $T(I)$ is a subset of the tropical prevariety of P. The Bieri-Groves theorem states that $V(I)$ (as an algebraic variety) and $T(I)$ (as a polyhedral complex) have the same dimension [3, 25].

Because tropical varieties and prevarieties are invariant under positive scaling, to determine that a variety in $(\mathbb{C} \setminus \{0\})^n$ is finite i.e. that it is zero-dimensional, it suffices to find a finite set of polynomials $P' \subseteq I$ such that $T(I) \subseteq \bigcap_{p'_i \in P'} T(p'_i) = \{0\}$. Then $V(I)$ is either finite or empty.

Our argument will be slightly more advanced, as we will also use that the cone over a tropical variety is a linear subspace of $\mathbb{R}^n$.

Lemma 2.1. *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Then the non-negative span S of $T(I)$ is a linear subspace of $\mathbb{R}^n$.*

Proof. The set $T(I)$ is the union of finitely many rational polyhedral cones in $\mathbb{R}^n$. This holds even when I is not homogeneous. See [12] for a proof. Consequently, the non-negative span of $T(I)$ is a rational polyhedral cone S.

Suppose for contradiction that S was not a linear space. Then S is a full-dimensional cone inside $\text{span}_{\mathbb{R}}(S)$ and has a rational supporting hyperplane H with $S \setminus H \neq \emptyset$. Therefore we may define a linear map $\pi : \mathbb{R}^n \to \mathbb{R}$ given by a matrix with integral coefficients with $\pi(S) = \mathbb{R}_{\geq 0}$.

The final step of the proof requires properties of balancing of tropical varieties. We refer to the upcoming book [17] for definitions and theorems. By [17, Theorem 3.3.6] $T(I)$ is balanced. Therefore the graph of π on $T(I)$ is balanced. By [17, Lemma 3.6.3] the projection $\pi(T(I))$ of the graph is balanced. However, $\pi(S) = \mathbb{R}_{\geq 0}$ implies that $\pi(T(I)) = \mathbb{R}_{\geq 0}$, which cannot be balanced. □

For more extensive background on tropical varieties see [12] or [17]. Similar use of tropical prevarieties is made in related problems in [9, 10, 14, 7, 8], where different versions of the lemma above appeared. We should also note that closely related techniques are used in proving a finiteness result in the five body problem in [2], although with different notation and vocabulary.

3. Finiteness

We fix a configuration of $N-1$ planar points $q_i \in \mathbb{R}^2$ with real masses $m_1, \ldots, m_{N-1}$, and consider the addition of q_N with mass m_N.

The main result of this paper is the following theorem:

Theorem 3.1. *For each fixed configuration in the* $(N-1)$*-body problem with nonzero masses, there are at most finitely many relative equilibria points for an additionally given (possibly zero mass) body, resulting in a nonzero total mass, for any potential function*

$$U = \begin{cases} \sum_{i<j} \frac{m_i m_j}{r_{ij}^{D-2}} & D > 2 \\ \sum_{i<j} m_i m_j \ln r_{ij} & D = 2 \end{cases}$$

where D *is a positive integer greater than or equal to* 2*.*

To prove the theorem we need to study equations in the r_{ij} and m_i which are known to be satisfied for central configurations. One such set of equations is the Albouy-Chenciner equations [1]; a short derivation is given in [9]. In the following we define an asymmetric version of these discovered by Gareth Roberts – see also [8]. We require that the total mass $\sum_{i=1}^{N} m_i \neq 0$ because this is a necessary assumption in deriving the Albouy-Chenciner equations.

Let $S_{ik} = r_{ik}^{-D} - 1$ for $k \neq i$ and $S_{ii} = 0$, and let $A_{ijk} = -r_{jk}^2 + r_{ik}^2 + r_{ij}^2$ for $i \neq j$. Note that $A_{iji} = 0$ and $A_{ijj} = 2r_{ij}^2$. For $i \neq j$ the asymmetric Albouy-Chenciner equations for N bodies can be written as

$$f_{ij} = \sum_{k=1}^{N} m_k S_{ik} A_{ijk} = 0 \tag{4}$$

In the formulas above we let $r_{ij} = r_{ji}$ if $i > j$ and will consider r_{iN} as variables and all other r_{ij} and m_i as parameters.

Because $x_1, \ldots, x_N \in \mathbb{R}^2$, the three-dimensional volume of the convex hull of any four of the points is zero. Therefore their Cayley-Menger determinant is also zero [4]. We write the Cayley-Menger determinant for the points 1, 2, 3, and N, emphasizing the dependence upon the variables r_{1N}, r_{2N}, and r_{3N}:

$$C(1,2,3,N) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{1N}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{2N}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{3N}^2 \\ 1 & r_{1N}^2 & r_{2N}^2 & r_{3N}^2 & 0 \end{vmatrix} =$$

$$-2(r_{23}^2)r_{1N}^4 - 2(r_{13}^2)r_{2N}^4 - 2(r_{12}^2)r_{3N}^4 +$$
$$2(-r_{12}^2 + r_{13}^2 + r_{23}^2)r_{1N}^2 r_{2N}^2 + 2(r_{12}^2 - r_{13}^2 + r_{23}^2)r_{1N}^2 r_{3N}^2 + 2(r_{12}^2 + r_{13}^2 - r_{23}^2)r_{2N}^2 r_{3N}^2 +$$
$$2((r_{12}^2 + r_{13}^2 - r_{23}^2)r_{23}^2)r_{1N}^2 + 2((r_{12}^2 - r_{13}^2 + r_{23}^2)r_{13}^2)r_{2N}^2 + 2((-r_{12}^2 + r_{13}^2 + r_{23}^2)r_{12}^2)r_{3N}^2$$
$$-2(r_{12}^2 r_{13}^2 r_{23}^2)$$

The Newton polytope of this polynomial (again thinking of only r_{1N}, r_{2N}, and r_{3N} as variables) is a simplex, with vertices $(4,0,0)$, $(0,4,0)$, $(0,0,4)$, and $(0,0,0)$. These vertices correspond respectively to the first three terms and the last term in the polynomial above.

Let us denote the set of Cayley-Menger determinants involving the Nth point as $\mathcal{C}_N$, i.e.

$$\mathcal{C}_N = \{C(i,j,k,N) | 1 \leq i < j < k < N\}$$

In a slight abuse of language, we will refer to the ray generated by a vector v as 'the ray v'.

We will denote the vector of all ones as $\bar{1}$, that is $\bar{1} = (1,\ldots,1) = \sum_{i=1}^{N-1} e_i$, where e_i is the unit vector in the i^{th} coordinate direction. Note that e_1 corresponds to the variable r_{1N}, e_2 to r_{2N}, et cetera.

Lemma 3.2. *The tropical prevariety $\bigcap_{C \in \mathcal{C}_N} T(C) \subseteq \mathbb{R}^{N-1}$ defined by $\mathcal{C}_N$ consists of the origin, the rays $-e_i$, the ray $\bar{1}$, cones $-ae_i - be_j$ with $a, b > 0$, and cones $b\bar{1} - ae_i$ with $b > 0$, $a \geq 0$.*

Proof. The membership of the rays in the tropical prevariety is clear since the Newton polytopes of the polynomials in $\mathcal{C}_N$ are the simplices described above.

The cones are also straightforward to verify:

The cones of the form $-ae_i - be_j$ will be maximized on an edge of $NP(C(i,j,k,N))$, for any k, a 2-face for the Newton polytopes of polynomials $C(i,k,l,N)$ or $C(j,k,l,N)$, and the entire simplex for $NP(C(k,l,m,N))$ where $i,j \notin \{k,l,m\}$.

The cones of the form $b\bar{1} - ae_i$ will be maximized on an edge of $NP(C(i,j,k,N))$, and on a 2-face for $NP(C(j,k,l,N))$.

Now we need to show that no other vectors are in the tropical prevariety.

We first prove that vectors with two or more negative weights cannot be in the tropical prevariety unless they have form $-ae_i - be_j$ with $a, b > 0$. In the other case, after projecting to three coordinates, the vector is $-ae_i - be_j +$

ce_k with $c \neq 0$. This vector will attain a unique maximum at a vertex of $NP(C(i,j,k,N))$ and can therefore not be in the tropical prevariety.

If there is a single negative weight, then on a simplex $NP(C(i,j,k,N))$ involving the negative weight variable the other two weights must be equal and non-negative. Then we can consider other simplices involving the negative weight and one of the two equal weights. The remaining weight must equal the non-negative weight. If the equal weights are zero, we obtain a ray $-e_i$. If the equal weights are positive, then the weight vector must be in a cone $b\bar{1} - ae_i$ with $a > b > 0$.

If there are no negative weights, then the vector is either the origin or has at least one positive weight. If there is a positive weight, consider the maximum weight. On any simplex $NP(C(i,j,k,N))$ involving the maximum weight, there must be another weight equal to it. This means there can be at most one weight less than the maximum. This means the vector is of the form $b\bar{1} - ae_i$ with $b > a \geq 0$.

The cones $b\bar{1} - ae_i$ with $b > a \geq 0$ and $b\bar{1} - ae_i$ with $a > b > 0$ are included in cones of the form $b\bar{1} - ae_i$ with $b > 0$, $a \geq 0$. □

We denote the set of asymmetric Albouy-Chenciner polynomials $\{p_{N1}, p_{N2}, \ldots, p_{N(N-1)}\}$ by $\mathcal{P}$. The polynomials p_{Nj} are defined as the numerators of the rational functions (Laurent polynomials) f_{Nj} given in equation (4).

Lemma 3.3. *The intersection of the tropical prevariety of $\mathcal{C}_N$, the tropical variety of $\mathcal{P}$, and the halfspace $\{w \in \mathbb{R}^{N-1} | w \cdot \bar{1} \leq 0\}$ is the origin.*

Proof. By Lemma 3.2, we need only consider the following cases. For convenience we will calculate with the Laurent polynomials f_{Ni}, rather than the polynomials p_{Ni}, which does not affect the computation of the tropical prevariety.

For a weight vector $w = -e_i, i \neq N$ and $k \neq N$, we have

$$\mathrm{in}_w(S_{kN}) = \mathrm{in}_w(r_{kN}^{-D} - 1) = \begin{cases} r_{kN}^{-D} & \text{of } w\text{-degree } D \quad \text{if } k = i \\ r_{kN}^{-D} - 1 & \text{of } w\text{-degree } 0 \quad \text{if } k \neq i \end{cases}$$

$$\mathrm{in}_w(A_{Nik}) = \mathrm{in}_w(-r_{ik}^2 + r_{kN}^2 + r_{iN}^2) = \begin{cases} 2r_{kN}^2 & \text{of } w\text{-degree -2} \quad \text{if } k = i \\ r_{kN}^2 - r_{ik}^2 & \text{of } w\text{-degree } 0 \quad \text{if } k \neq i \end{cases}$$

If $D > 2$ then $\deg_w(m_k S_{kN} A_{Nik})$ is largest when $k = i$, and hence

$$\mathrm{in}_w(f_{Ni}) = \mathrm{in}_w(\sum_{k=1}^{N-1} m_k S_{kN} A_{Nik}) = \mathrm{in}_w(m_i S_{iN} A_{Nii}) = 2m_i r_{iN}^{2-D},$$

a monomial.

If $D = 2$ then the case of $w = -e_i$ is a little more complicated, and we need to construct a monomial from a combination of $\mathrm{in}_w(f_{iN})$ with $\mathrm{in}_w(f_{jN})$ for $j \neq i$. For $D = 2$ we have for i and $j \neq i$:

$$\mathrm{in}_w(f_{Ni}) = \mathrm{in}_w(\sum_{k=1}^{N-1} m_k S_{kN} A_{Nik}) = 2m_i + \sum_{k \neq i} m_k S_{kN}(-r_{ik}^2 + r_{kN}^2)$$

$$\mathrm{in}_w(f_{Nj}) = \mathrm{in}_w(\sum_{k=1}^{N-1} m_k S_{kN} A_{Njk}) = m_i r_{iN}^{-2}(-r_{ij}^2 + r_{jN}^2),$$

because A_{Njk} has w-degree 0 for all k, whereas S_{kN} has largest w-degree when $k = i$. In that case $\mathrm{in}_w(S_{kN}) = r_{iN}^{-2}$.

We can combine the above initial forms to create a monomial:

$$m_i \mathrm{in}_w(f_{Ni}) - \sum_{k \neq i} m_k r_{iN}^2 S_{kN} \mathrm{in}_w(f_{Nk}) = 2m_i^2.$$

For a weight vector $w = -ae_i - be_j$ with $a, b > 0$ and $i \neq j$, we can assume without loss of generality that $a \geq b$. We have

$$\mathrm{in}_w(S_{kN}) = \mathrm{in}_w(r_{kN}^{-D} - 1) = \begin{cases} r_{kN}^{-D} & \text{of } w\text{-degree } Da & \text{if } k = i \\ r_{kN}^{-D} & \text{of } w\text{-degree } Db & \text{if } k = j \\ r_{kN}^{-D} - 1 & \text{of } w\text{-degree } 0 & \text{if } k \notin \{i, j\} \end{cases}$$

$$\mathrm{in}_w(A_{Njk}) = \mathrm{in}_w(-r_{jk}^2 + r_{kN}^2 + r_{jN}^2) = \begin{cases} -r_{jk}^2 & \text{of } w\text{-degree } 0 & \text{if } k = i \\ 2r_{jN}^2 & \text{of } w\text{-degree -2b} & \text{if } k = j \\ r_{kN}^2 - r_{jk}^2 & \text{of } w\text{-degree } 0 & \text{if } k \notin \{i, j\} \end{cases}$$

Hence $\deg_w(m_k S_{kN} A_{Njk})$ is uniquely maximized at $k = i$ and

$$\mathrm{in}_w(f_{Nj}) = \mathrm{in}_w(\sum_{k=1}^{N-1} m_k S_{kN} A_{Njk}) = \mathrm{in}_w(m_i S_{iN} A_{Nji}) = -m_i r_{iN}^{-D} r_{ij}^2.$$

For a weight vector $w = b\bar{1} - ae_i$ with $a > b > 0$, which will cover our last case, we choose $j \neq i$ and we have

$$\mathrm{in}_w(S_{kN}) = \mathrm{in}_w(r_{kN}^{-D} - 1) = \begin{cases} r_{kN}^{-D} & \text{of } w\text{-degree } (a - b)D > 0 & \text{if } k = i \\ 1 & \text{of } w\text{-degree } 0 & \text{if } k \neq i \end{cases}$$

$$\mathrm{in}_w(A_{Njk}) = \mathrm{in}_w(-r_{jk}^2 + r_{kN}^2 + r_{jN}^2) = \begin{cases} r_{jN}^2 & \text{of } w\text{-degree } 2b \quad \text{if } k = i \\ r_{kN}^2 + r_{jN}^2 & \text{of } w\text{-degree } 2b \quad \text{if } k \neq i \end{cases}$$

Hence

$$\mathrm{in}_w(f_{Nj}) = \mathrm{in}_w(\sum_{k=1}^{N-1} m_k S_{kN} A_{Njk}) = \mathrm{in}_w(m_i S_{iN} A_{Nji}) = m_i r_{iN}^{-D} r_{jN}^2$$

a nonzero monomial. □

Proof of Theorem 3.1: This follows immediately from Lemma 3.3 and Lemma 2.1.

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