Representations of Lie algebras in prime characteristic

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Abstract

The aim of these lectures is to give a survey on the representation theory of Lie algebras of reductive groups in prime characteristic. This theory is quite different from the corresponding theory in characteristic 0. For example, in prime characteristic all simple modules are finite dimensional. On the other hand, there is in most cases no classification of these simple modules. There has been major progress in this area in the last few years, mostly related to Premet's proof (from 1995) of the Kac-Weisfeiler conjecture (from 1971).

The first four sections discuss the representation theory of general (restricted) Lie algebras in prime characteristic as well as some special aspects in the cases of unipotent and solvable Lie algebras. The rest of the text then deals more specifically with Lie algebras of reductive groups.

Throughout K will be an algebraically closed field, $\mathrm{char}(K) = p > 0$. If $\mathfrak{g}$ is a Lie algebra over K, we will let $U(\mathfrak{g})$ denote its universal enveloping algebra and $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$. All Lie algebras over K will be assumed to be finite dimensional.

1. Finiteness

1.1. The representation theory of Lie algebras in prime characteristic has certain features that make it completely different from that of Lie algebras in characteristic 0. This is very well illustrated by the following theorem:

THEOREM. *Let $\mathfrak{g}$ be a Lie algebra over K.*
a) *Each irreducible representation of $\mathfrak{g}$ is finite dimensional.*
b) *There exists a positive integer $M(\mathfrak{g})$ such that every irreducible representation of $\mathfrak{g}$ has dimension less than $M(\mathfrak{g})$.*

This result should be contrasted with the situation in characteristic 0 where already the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ has both infinite dimensional irreducible representations and finite dimensional irreducible representations of arbitrarily large dimension, see [**24**], §7, Exercises 3 and 7.

This theorem appears for the first time in [**7**], 5.1. The fact that the dimensions of the finite dimensional irreducible representations are bounded was found independently in [**56**]; the footnote on the first page of [**56**] makes clear that also Jacobson was aware of this fact.

1.2. Theorem 1.1 can be easily deduced from the following result contained in [**25**]:

THEOREM. *The algebra $U(\mathfrak{g})$ is a finitely generated $Z(\mathfrak{g})$-module and $Z(\mathfrak{g})$ is a finitely generated K-algebra.*

1.3. Let us show that Theorem 1.2 implies Theorem 1.1. Suppose that

$$U(\mathfrak{g}) = \sum_{i=1}^{r} Z(\mathfrak{g})u_i.$$

Let V be a simple $U(\mathfrak{g})$-module. Consider $v \in V$, $v \neq 0$. Then

$$V = U(\mathfrak{g}).v = \sum_{i=1}^{r} Z(\mathfrak{g})u_i.v$$

So the module V is finitely generated over $Z(\mathfrak{g})$. Hence, since $Z(\mathfrak{g})$ is Noetherian, there exists a maximal $Z(\mathfrak{g})$-submodule $V' \subset V$. Thus

$$V/V' \simeq_{Z(\mathfrak{g})} Z(\mathfrak{g})/\mathfrak{m}$$

for some maximal ideal $\mathfrak{m}$ of $Z(\mathfrak{g})$. Hence $\mathfrak{m}V \subsetneqq V$. But $\mathfrak{m}V$ is a $U(\mathfrak{g})$-module so $\mathfrak{m}V = 0$. Therefore $Z(\mathfrak{g})$ acts on V via $Z(\mathfrak{g})/\mathfrak{m} = K$. This proves part (a) of the theorem and we see that $r+1$ suffices as our bound, $M(\mathfrak{g})$.

1.4. Let $U_\varepsilon(\mathfrak{g})$ be the quantised enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$ at a root of unity, see [**9**]. Then the representation theory of this algebra (in characteristic 0) has many similarities with that of Lie algebras in prime characteristic. A first indication for this phenomenon is the fact that the two theorems above generalise: The algebra centre of $U_\varepsilon(\mathfrak{g})$, denoted $Z(U_\varepsilon(\mathfrak{g}))$, is a finitely generated $\mathbf{C}$-algebra and $U_\varepsilon(\mathfrak{g})$ is finitely generated over $Z(U_\varepsilon(\mathfrak{g}))$, see [**9**], §3. The argument above shows then that also Theorem 1.1 generalises to $U_\varepsilon(\mathfrak{g})$.

1.5. Example 1. Define

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{g} = Kh + Kx$ with the usual commutator. Note that $[h,x] = x$. We have

$$\operatorname{ad}(x)^2 = 0 \qquad \text{and} \qquad \operatorname{ad}(h)^2 = \operatorname{ad}(h),$$

hence

$$\operatorname{ad}(x)^p = 0 \qquad \text{and} \qquad \operatorname{ad}(h)^p = \operatorname{ad}(h).$$

Since $\operatorname{ad}(y)^p(u) = y^p u - u y^p$ for all $u, y \in U(\mathfrak{g})$ we see $x^p \in Z(\mathfrak{g})$ and $h^p - h \in Z(\mathfrak{g})$. Hence $K[x^p, h^p - h] \subset Z(\mathfrak{g})$. In particular we have

$$U(\mathfrak{g}) = \sum_{i,j<p} Z(\mathfrak{g})x^i h^j.$$

Using the commutation relations in $U(\mathfrak{g})$

$$hx^i = x^i(h+i) \quad \text{and} \quad xh^j = (h-1)^j x \tag{1}$$

one can show $K[x^p, h^p - h] = Z(\mathfrak{g})$, but we are not going to use this stronger result.

Let M be an irreducible $\mathfrak{g}$-module. Then x^p acts by a scalar, a^p, on M. Hence

$$(x-a)^p.M = (x^p - a^p).M = 0.$$

Case I : $a = 0$. We see that $M_0 = \{m \in M \mid x.m = 0\} \neq 0$ and that M_0 is a submodule of M, so $M_0 = M$ and $x.M = 0$. So M is a simple module for the commutative Lie algebra

$\mathfrak{g}/Kx$. Therefore $\dim M = 1$ and h has to act by a scalar on M. (Conversely, given any scalar $b \in K$, we get a one dimensional simple $\mathfrak{g}$–module where x acts as 0 and h as b.)
Case II : $a \neq 0$. There exists $0 \neq e_0 \in M$ such that $x.e_0 = ae_0$. There exists $b \in K$ such that

$$(h^p - h)_{|M} = b^p.$$

Set $e_i = h^i e_0$. Then $e_p = b^p e_0 + e_1$ and by (1) we have

$$x.e_i = (h-1)^i a e_0 = a e_i - i a e_{i-1} + \cdots . \qquad (2)$$

It follows that the span of $e_0, e_1, \ldots, e_{p-1}$ is stable under x and h, hence a submodule of M. So the irreducibility of M shows that M is generated by $e_0, e_1, \ldots, e_{p-1}$. Using (2) one can show the elements $e_0, \ldots, e_i$ are linearly independent for all $i < p$ by induction on i. This implies then that $e_0, e_1, \ldots, e_{p-1}$ is a basis of M, in particular $\dim M = p$.

(Here is the induction argument: The claim holds for $i = 0$ because $e_0 \neq 0$. For $i > 0$ assume that e_i is a linear combination of the e_j with $j < i$. Then $(x-a)e_i$ is a linear combination of the $(x-a)e_j$ with $j < i$. Each $(x-a)e_j$ is by (2) a linear combination of the e_h with $h < j$, so $(x-a)e_i$ is a linear combination of the e_j with $j < i-1$. On the other hand, $(x-a)e_i$ is by (2) equal to $-iae_{i-1}$ plus a linear combination of the e_j with $j < i-1$. Now $a \neq 0$ and $0 < i < p$ yield that e_{i-1} is a linear combination of the e_j with $j < i-1$ contradicting the induction hypothesis.)

Conversely, given $a, b \in K$ with $a \neq 0$, we can find a simple $\mathfrak{g}$–module M of dimension p with basis $e_0, e_1, \ldots, e_{p-1}$ such that x and h act as described above.

Note that our Lie algebra $\mathfrak{g}$ is solvable. So this example shows also that Lie's Theorem does not generalise to prime characteristic. Our $\mathfrak{g}$ is actually the standard example for this fact, e.g., in [**24**], Exercise 3 in §4.

2. Restricted Lie Algebras

Before we go on we make the convention that from now on all $\mathfrak{g}$–modules are assumed to be finite dimensional.

2.1. The restricted Lie algebras (also called 'Lie p–algebras') form an important class of Lie algebras in prime characteristic including all those that we (in these lectures) are really interested in. I shall first give an ad hoc description of these objects and then state the 'real' definition.

Suppose $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n(K) = M_n(K)$. We say $\mathfrak{g}$ is *restricted* if for all $x \in \mathfrak{g}$ we have $x^p \in \mathfrak{g}$ [where the pth power is taken in $\mathfrak{gl}_n(K) = M_n(K)$].
Example. If $G \leq GL_n(K)$ is an algebraic subgroup then $\mathfrak{g} = \mathrm{Lie}(G)$ is restricted.

Assume that $\mathfrak{g} \subset \mathfrak{gl}_n(K)$ is restricted. We have a notational problem: There is a pth power of $x \in \mathfrak{g}$ in $M_n(K)$ and a pth power of x in $U(\mathfrak{g})$. In order to distinguish them we will write $x^{[p]}$ for the pth power in $M_n(K)$ and reserve the notation x^p for the pth power in $U(\mathfrak{g})$.

In Example 1 we used that $\mathrm{ad}(y)^p(u) = y^p u - u y^p$ for all y and u in an associative algebra over K. Applying this to $M_n(K)$ we get $\mathrm{ad}(x)^p = \mathrm{ad}(x^{[p]})$ for all $x \in \mathfrak{g}$. This holds first for $\mathrm{ad}(x)$ acting on $M_n(K)$, but then also for $\mathrm{ad}(x)$ acting on $\mathfrak{g}$ and then finally also for $\mathrm{ad}(x)$ acting on $U(\mathfrak{g})$. In other words, we get for all $x \in \mathfrak{g}$

$$x^p - x^{[p]} \in Z(\mathfrak{g}). \qquad (1)$$

Define $\xi : \mathfrak{g} \longrightarrow Z(\mathfrak{g})$ by $x \longmapsto x^p - x^{[p]}$.

Lemma. *The map ξ is semilinear. That is, for all $a \in K$ and $x, y \in \mathfrak{g}$,*

$$\xi(x+y) = \xi(x) + \xi(y), \qquad \xi(ax) = a^p \xi(x).$$

Here the proof of the second equality is trivial. The first one requires more work. It says that $(x+y)^p - x^p - y^p = (x+y)^{[p]} - x^{[p]} - y^{[p]}$. One has to show that $(u+v)^p - u^p - v^p$ for all u and v in an associative algebra can be expressed in terms of iterated commutators of u and v, see [**26**], Formula (63) in Chapter V. Then one has to use that one gets in our situation the same commutators in $U(\mathfrak{g})$ and in $M_n(K)$.

2.2. We use this lemma as a motivation for the following abstract definition:
Definition: A *restricted Lie algebra* over K is a Lie algebra $\mathfrak{g}$ over K with a map $\mathfrak{g} \longrightarrow \mathfrak{g}$ sending $x \longmapsto x^{[p]}$ such that $\xi(x) = x^p - x^{[p]} \in Z(\mathfrak{g})$ for all $x \in \mathfrak{g}$ and such that $\xi : \mathfrak{g} \longrightarrow Z(\mathfrak{g})$ is semilinear. The map $x \longmapsto x^{[p]}$ is then called the *pth power map* of $\mathfrak{g}$.

One can show that each restricted Lie algebra over K in this abstract sense is isomorphic to some restricted $\mathfrak{g} \subset \mathfrak{gl}_n(K)$ as considered above. (Recall that we assume our Lie algebras to have finite dimension.) One can also check that the definition here is equivalent with the traditional definition, see [**26**], Definition 4 in Chapter V.

2.3. For the remainder of this section we will assume that $\mathfrak{g}$ is a restricted Lie algebra over K. Set $Z_0(\mathfrak{g}) = K[x^p - x^{[p]} \mid x \in \mathfrak{g}] \subset Z(\mathfrak{g})$. Let $\{x_1, x_2, \ldots, x_m\}$ be a basis of $\mathfrak{g}$. The semilinearity of ξ implies $Z_0(\mathfrak{g}) = K[\xi(x_1), \xi(x_2), \ldots, \xi(x_m)]$. The following proposition is therefore a consequence of the PBW theorem.

PROPOSITION. a) *The elements* $\xi(x_1), \xi(x_2), \ldots, \xi(x_m)$ *are algebraically independent generators for* $Z_0(\mathfrak{g})$.
b) *The algebra* $U(\mathfrak{g})$ *is free over* $Z_0(\mathfrak{g})$ *with basis*

$$\{x_1^{a_1} x_2^{a_2} \ldots x_m^{a_m} \mid 0 \le a_i < p \text{ for all } i\}.$$

Remark. This implies Theorem 1.2 for restricted Lie algebras.

2.4. Let E be a simple $\mathfrak{g}$-module. Then for all $x \in \mathfrak{g}$ the element $\xi(x)$ acts by a scalar on E. This scalar can written as $\chi_E(x)^p$ for some $\chi_E(x) \in K$. The semilinearity of ξ yields now:

LEMMA ([**53**]). *For any simple* $\mathfrak{g}$*-module* E *we have* $\chi_E \in \mathfrak{g}^*$.

Definition: The functional χ_E is called the *p–character* of E. More generally if V is a $\mathfrak{g}$–module and $\chi \in \mathfrak{g}^*$ then we say V has p–character χ if and only if, for all $x \in \mathfrak{g}$,

$$(x^p - x^{[p]} - \chi(x)^p).V = 0.$$

2.5. Let M, M' be $\mathfrak{g}$–modules. One can show that, for all $m \in M, m' \in M', f \in M^*$, and $x \in \mathfrak{g}$,

$$\xi(x).(m \otimes m') = \xi(x).m \otimes m' + m \otimes \xi(x).m'$$

and

$$(\xi(x).f)(m) = -f(\xi(x).m).$$

It follows that if M has p–character χ and M' has p–character χ' then $M \otimes M'$ has p–character $\chi + \chi'$ and M^* has p–character $-\chi$.

2.6. The $\mathfrak{g}$–modules with p–character 0 correspond to the 'restricted representations' of $\mathfrak{g}$, i.e., to Lie algebra homomorphisms $\rho : \mathfrak{g} \to \mathrm{End}_K(V)$ with $\rho(x^{[p]}) = \rho(x)^p$ for all $x \in \mathfrak{g}$.

2.7. For $\chi \in \mathfrak{g}^*$ define

$$U_\chi(\mathfrak{g}) \equiv U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}).$$

Each $U_\chi(\mathfrak{g})$ is called a *reduced enveloping algebra* of $\mathfrak{g}$. (For $\chi = 0$ one usually calls $U_\chi(\mathfrak{g})$ the 'restricted enveloping algebra' of $\mathfrak{g}$; Jacobson originally called it the 'u–algebra' of $\mathfrak{g}$.)

The bijection

$$\{\mathfrak{g}\text{–modules}\} \longleftrightarrow \{U(\mathfrak{g})\text{–modules}\}$$

induces for each χ a bijection

$$\{\mathfrak{g}\text{–modules with } p\text{–character } \chi\} \longleftrightarrow \{U_\chi(\mathfrak{g})\text{–modules}\}.$$

2.8. Proposition 2.3 implies easily:

PROPOSITION. *If $\{x_1, x_2, \dots, x_m\}$ is a basis of $\mathfrak{g}$ then the algebra $U_\chi(\mathfrak{g})$ has basis*

$$\{x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} \mid 0 \le a_i < p \text{ for all } i\}.$$

We get in particular:

COROLLARY. *We have* $\dim U_\chi(\mathfrak{g}) = p^{\dim(\mathfrak{g})}$.

Remark. We have seen above that each simple $\mathfrak{g}$-module E has a p–character, hence is a simple $U_\chi(\mathfrak{g})$-module for some χ. Since

$$\dim(\mathrm{Hom}_{\mathfrak{g}}(U_\chi(\mathfrak{g}), E)) = \dim E, \tag{1}$$

there is a homomorphism

$$U_\chi(\mathfrak{g}) \twoheadrightarrow E^{\dim E},$$

so $\dim E \le \sqrt{p^{\dim \mathfrak{g}}}$. So we get an explicit value for a bound as in Theorem 1.1 (b).

2.9. Let $\widetilde{G} = \mathrm{Aut}_{\mathrm{res}}(\mathfrak{g})$, the group of all automorphisms τ of $\mathfrak{g}$ preserving the pth power map, i.e., with $\tau(x^{[p]}) = \tau(x)^{[p]}$ for all x. Then each $\tau \in \widetilde{G}$ induces an isomorphism of algebras

$$U_\chi(\mathfrak{g}) \xrightarrow{\sim} U_{\tau(\chi)}(\mathfrak{g})$$

where $\tau(\chi) = \chi \circ \tau^{-1}$. As a result the representation theory of $U_\chi(\mathfrak{g})$ depends only on the $\widetilde{G}$-orbit of χ.

If $\mathfrak{g} = \mathrm{Lie}(G)$ for some algebraic group G, then the adjoint representation of G is a homomorphism $G \to \mathrm{Aut}_{\mathrm{res}}(\mathfrak{g})$. Therefore the representation theory of $U_\chi(\mathfrak{g})$ depends only on the coadjoint G–orbit of χ.

2.10. Example 2. a) Let $\mathfrak{g} = K$ be the Lie algebra of the multiplicative group $G_m = \mathrm{GL}_1(K)$. Then $1^{[p]} = 1$ and $\mathfrak{g} = K.1$. Hence we can identify $U(\mathfrak{g})$ with the polynomial ring $K[t]$ in one variable t

$$U(\mathfrak{g}) \simeq K[t]$$

and get for each χ

$$U_\chi(\mathfrak{g}) \simeq K[t]/(t^p - t - \chi(1)^p). \tag{1}$$

Since the polynomial $t^p - t - \chi(1)^p$ is separable, we get $U_\chi(\mathfrak{g}) \simeq K \times \dots \times K$; so this algebra is semisimple for every $\chi \in \mathfrak{g}^*$.

One can generalise this example and show that $U_\chi(\mathfrak{h})$ is a commutative semisimple algebra if $\mathfrak{h}$ is the Lie algebra of a torus. (See the discussion of $\mathfrak{h}$–modules in 6.2.)

b) Let $\mathfrak{g} = K$ be the Lie algebra of the additive group G_a. We can embed G_a into $\mathrm{GL}_2(K)$ as set of all matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and identify thus $\mathfrak{g}$ as the set of all matrices $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$. This shows that $1^{[p]} = 0$ in $\mathfrak{g}$. Hence we have

$$U_\chi(\mathfrak{g}) \simeq K[t]/(t^p - \chi(1)^p) = K[t]/((t - \chi(1))^p). \tag{2}$$

This algebra has only one simple module, K, and is clearly not semisimple.

2.11. Example 2 shows that one Lie algebra ($\mathfrak{g} = K$) can have two different structures as a restricted Lie algebra and that then the reduced enveloping algebras $U_\chi(\mathfrak{g})$ can have completely different properties. On the other hand, in these two cases the representation theory of $U_\chi(\mathfrak{g})$ is more or less independent of χ. That changes immediately when we look at somewhat more complicated examples, such as that from Section 1:

Example 1. So we consider as before $\mathfrak{g} = Kh + Kx$ with $[h, x] = x$. It is a restricted subalgebra of $\mathfrak{gl}_2(K)$ satisfying $h^{[p]} = h$ and $x^{[p]} = 0$. If $\chi \in \mathfrak{g}^*$, then x^p acts as $\chi(x)^p$ on any $U_\chi(\mathfrak{g})$–module. So the computations in Section 1 show:
Case I : If $\chi(x) = 0$ then each simple $U_\chi(\mathfrak{g})$–module has dimension 1. Given a scalar $b \in K$, then the one dimensional module where h acts as multiplication by b (and x as 0) has p–character χ if and only if b satisfies $b^p - b - \chi(h)^p = 0$ (because $h^p - h - \chi(h)^p \in U(\mathfrak{g})$). This shows that there are exactly p one dimensional simple $U_\chi(\mathfrak{g})$–modules.
Case II : If $\chi(x) \neq 0$ then each simple $U_\chi(\mathfrak{g})$–module E has dimension p. Since $U_\chi(\mathfrak{g})$ maps onto $E^{\dim E}$ and both terms have dimension p^2, we see that $U_\chi(\mathfrak{g})$ is isomorphic to E^p as a module over itself. It follows that there is only one simple $U_\chi(\mathfrak{g})$–module and that $U_\chi(\mathfrak{g})$ is a semisimple (actually: a simple) algebra.

In the first case the Jacobson radical of $U_\chi(\mathfrak{g})$ is generated by x and has dimension $p^2 - p$, so, in particular, $U_\chi(\mathfrak{g})$ is not semisimple. The quotient of $U_\chi(\mathfrak{g})$ by its radical has dimension p; it is (as a module) the direct sum of the p simple $U_\chi(\mathfrak{g})$–modules.

2.12. For the quantised enveloping algebra $U_\varepsilon(\mathfrak{g})$ from 1.4 we have a similar picture: One defines a subalgebra Z_0 of the centre of $U_\varepsilon(\mathfrak{g})$ analogous to our $Z_0(\mathfrak{g})$, see [**9**], 3.3. The whole algebra $U_\varepsilon(\mathfrak{g})$ is then a free module of finite rank over Z_0 with an explicit basis, similar to the one in Proposition 2.3.b.

At the next step there is a minor modification: Our $Z_0(\mathfrak{g})$ is a polynomial algebra in $\dim(\mathfrak{g})$ variables. Therefore the set of all algebra homomorphisms $Z_0(\mathfrak{g}) \to K$ can be identified with $K^{\dim(\mathfrak{g})}$. Actually, we identify it with $\mathfrak{g}^*$ such that $\chi \in \mathfrak{g}^*$ corresponds to the homomorphism with $\xi(x) \mapsto \chi(x)^p$ for all $x \in \mathfrak{g}$.

The algebra Z_0 is a localisation of the form $\mathbf{C}[x_1, x_2, \dots, x_m, t_1, t_2, \dots, t_n, t_1^{-1}, t_2^{-1}, \dots, t_n^{-1}]$ of a polynomial algebra in variables x_i and t_j. Now the set Ω of all algebra homomorphisms $Z_0 \to \mathbf{C}$ can be identified with $\mathbf{C}^m \times (\mathbf{C}^\times)^n$ where $\mathbf{C}^\times$ denotes the set of non-zero complex numbers. So Ω looks like the 'big cell' in a suitable semisimple algebraic group over $\mathbf{C}$. Actually, one identifies Ω canonically with an unramified cover of that big cell.

If E is a simple $U_\varepsilon(\mathfrak{g})$–module, then Z_0 acts on E via some character $\chi_E \in \Omega$. That is the quantum analogue of a p–character.

Instead of the action on $\mathfrak{g}^*$ of the automorphism group from 2.9 one now has the 'quantum coadjoint action' on Ω of a group $\widetilde{G}$ (more complicated to define). Conjugate elements in Ω lead then (as in 2.9) to equivalent representation theories. (For all this consult [**9**] and [**11**].)

3. Unipotent Lie Algebras

3.1. Definition: Let $\mathfrak{g}$ be a restricted Lie algebra. We call $\mathfrak{g}$ *unipotent* (or 'p–nilpotent') if, for all $x \in \mathfrak{g}$, there exists $r > 0$ such that $x^{[p^r]} = 0$, where $x^{[p^r]}$ denotes the pth power map iterated r times.
Example. The restricted Lie algebra from Example 2b is unipotent. More generally, if G is a unipotent algebraic group then $\mathfrak{g} = \mathrm{Lie}(G)$ is unipotent. (Well, we can suppose that G is a closed subgroup of some $\mathrm{GL}_n(K)$ contained in the subgroup of upper triangular matrices with all diagonal entries equal to 1. Then $\mathfrak{g}$ can be identified with a Lie subalgebra of

$\mathfrak{gl}_n(K)$ contained in the Lie subalgebra of all strictly upper triangular matrices. Under this identification the pth power map in $\mathfrak{g}$ corresponds to taking the pth power as a matrix.)
Remark. One can identify the category of restricted Lie algebras over K with the category of certain (infinitesimal) group schemes over K, see [**13**], Ch. II, §7, n° 4. Then a restricted Lie algebra is unipotent (in the sense of the definition above) if and only if the corresponding group scheme is unipotent, see [**13**], Ch. IV, §2, Corollaire 2.13.

3.2. Proposition. *If $\mathfrak{g}$ is unipotent then the trivial $\mathfrak{g}$–module K is the only simple $U_0(\mathfrak{g})$-module (up to isomorphism).*

Indeed, in the algebra $U_0(\mathfrak{g})$ we have $x^{[p]} = x^p$ for all $x \in \mathfrak{g}$, and so $x^{[p^r]} = x^{p^r}$. Hence $U_0(\mathfrak{g})\mathfrak{g}$ is a nilpotent ideal and the claim follows.
Remark. This is an analogue to the fact for algebraic groups that a unipotent group has a unique irreducible module.

3.3. Theorem. *If $\mathfrak{g}$ is unipotent then each $U_\chi(\mathfrak{g})$ has only one simple module (up to isomorphism).*

Proof. Let E and E' be simple $U_\chi(\mathfrak{g})$–modules. Then $\mathrm{Hom}_K(E, E') \simeq E^* \otimes E'$ is a $U_0(\mathfrak{g})$-module. Thus we have $K \subset \mathrm{Hom}_K(E, E')$ as a $\mathfrak{g}$-submodule. Therefore $\mathrm{Hom}_\mathfrak{g}(E, E') \neq 0$ and so, by Schur's lemma, $E \simeq E'$.
Remark. The argument used above can be found in [**55**] where Zassenhaus attributes the idea to Whitehead and Witt. It is used there to prove a more general result, valid for all nilpotent Lie algebras. It reduces to the theorem here in the unipotent case.

The article [**55**] is otherwise mainly a survey of the earlier paper [**54**] by Zassenhaus, which can be regarded as the starting point of the representation theory of Lie algebras in prime characteristic.

3.4. Corollary. *Let $\mathfrak{g}$ be unipotent. Let $\chi \in \mathfrak{g}^*$ and E the simple $U_\chi(\mathfrak{g})$-module. If $\dim E = 1$ then $U_\chi(\mathfrak{g})$ is indecomposable as a $\mathfrak{g}$-module. In particular every projective $U_\chi(\mathfrak{g})$-module is free over $U_\chi(\mathfrak{g})$.*

Proof. As $\dim E = 1$ we have by 2.8(1) an isomorphism of $\mathfrak{g}$–modules

$$U_\chi(\mathfrak{g})/\operatorname{rad} U_\chi(\mathfrak{g}) \simeq E.$$

4. Induction

4.1. Let $\mathfrak{g}$ be a restricted Lie algebra and $\mathfrak{s} \subset \mathfrak{g}$ a restricted Lie subalgebra. Let $\chi \in \mathfrak{g}^*$. We have an algebra map induced by inclusion

$$U_\chi(\mathfrak{s}) \equiv U_{\chi|_\mathfrak{s}}(\mathfrak{s}) \longrightarrow U_\chi(\mathfrak{g})$$

which, by Proposition 2.8, is injective.

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of $\mathfrak{g}$ such that $\{x_1, x_2, \ldots, x_l\}$ (for some $l \leq n$) is a basis of $\mathfrak{s}$. Using Proposition 2.8 again, we get:

Proposition. *The algebra $U_\chi(\mathfrak{g})$ is a free $U_\chi(\mathfrak{s})$–module with basis*

$$\{x_{l+1}^{a_{l+1}} x_{l+2}^{a_{l+2}} \ldots x_n^{a_n} \mid 0 \leq a_i < p \text{ for all } i\}.$$

4.2. If M is a $U_\chi(\mathfrak{s})$–module then

$$\operatorname{ind}_\chi(M) \equiv U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{s})} M \tag{1}$$

is a $U_\chi(\mathfrak{g})$–module, called an *induced module.* It is clear that $M \mapsto \operatorname{ind}_\chi(M)$ is a functor from $\{ U_\chi(\mathfrak{s})\text{–modules} \}$ to $\{ U_\chi(\mathfrak{g})\text{–modules} \}$. By Proposition 4.1 this functor is exact and satisfies

$$\dim(\operatorname{ind}_\chi(M)) = p^{\dim(\mathfrak{g}/\mathfrak{s})} \dim M. \tag{2}$$

More explicitly, if $m_1, m_2, \dots, m_r$ is a basis of M and if the x_i are as in Proposition 4.1, then all $x_{l+1}^{a_{l+1}} x_{l+2}^{a_{l+2}} \dots x_n^{a_n} \otimes m_j$ are a basis of $\operatorname{ind}_\chi(M)$.

Furthermore induction satisfies 'Frobenius reciprocity', that is:

Lemma. *If V is a $U_\chi(\mathfrak{g})$–module and M a $U_\chi(\mathfrak{s})$–module then we have a functorial isomorphism*

$$\operatorname{Hom}_\mathfrak{g}(\operatorname{ind}_\chi(M), V) \xrightarrow{\sim} \operatorname{Hom}_\mathfrak{s}(M, V).$$

(One maps any $\psi : \operatorname{ind}_\chi(M) \to V$ to $\overline{\psi} : M \to V$ with $\overline{\psi}(m) = \psi(1 \otimes m)$. And one maps any $\varphi : M \to V$ to $\widetilde{\varphi} : \operatorname{ind}_\chi(M) \to V$ with $\widetilde{\varphi}(u \otimes m) = u.\varphi(m)$.)

Remark. As mentioned in Remark 3.1, we can regard a restricted Lie algebra $\mathfrak{g}$ as a group scheme. Its representations as a group scheme correspond to the $U_0(\mathfrak{g})$–modules. A restricted Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ can then be regarded as a closed subscheme of $\mathfrak{g}$. Now you should be warned that the usual induction functor for group schemes (as in [**29**], I.3.3) does *not* correspond to our ind_0. The induction functor for group schemes is right adjoint to the restriction functor, while the lemma here says that our induction functor is left adjoint to the restriction functor. Our ind_0 corresponds to what is called *coinduction* in the representation theory of (finite) group schemes, see [**29**], I.8.14.

4.3. Example 3. Let $\mathfrak{g}$ be the Lie algebra of all strictly upper triangular (3×3)–matrices over K. So this is a three dimensional restricted Lie algebra with basis

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $[x,y] = z$ and $[x,z] = [y,z] = 0$. The pth power map is given by $x^{[p]} = y^{[p]} = z^{[p]} = 0$. This shows that $\mathfrak{g}$ is unipotent.

Let $\chi \in \mathfrak{g}^*$. Because z is central in $\mathfrak{g}$ it has to act by a scalar on each simple $U_\chi(\mathfrak{g})$–module E. Since $z^p - \chi(z)^p$ annihilates E, it follows that this scalar has to be equal to $\chi(z)$.

Case I : Suppose that $\chi(z) = 0$. Then z annihilates E and E is a simple module for the two dimensional Lie algebra $\mathfrak{g}/Kz$. Because $\mathfrak{g}/Kz$ is commutative we get $\dim(E) = 1$. So also x and y have to act as scalars; these scalars have to be equal to $\chi(x)$ and $\chi(y)$ respectively. It then follows that each $u \in \mathfrak{g}$ acts as multiplication by $\chi(u)$ on E.

Case II : Suppose that $\chi(z) \neq 0$. The elements y and z span a two dimensional restricted Lie subalgebra $\mathfrak{s} = Ky + Kz$ of $\mathfrak{g}$. Each simple $\mathfrak{s}$–module has dimension 1 since $\mathfrak{s}$ is commutative. Furthermore y has to act as multiplication by $\chi(y)$ on each simple $U_\chi(\mathfrak{s})$–module (by the argument applied above to z). Therefore the unique (up to isomorphism) simple $U_\chi(\mathfrak{s})$–module is K_χ, that is K where each $u \in \mathfrak{s}$ acts as multiplication by $\chi(u)$.

Since our simple $U_\chi(\mathfrak{g})$–module E has to contain a simple $U_\chi(\mathfrak{s})$–submodule, we have $\operatorname{Hom}_\mathfrak{s}(K_\chi, E) \neq 0$, hence by Frobenius reciprocity $\operatorname{Hom}_\mathfrak{g}(\operatorname{ind}_\chi(K_\chi), E) \neq 0$. Therefore E is a homomorphic image of $\operatorname{ind}_\chi(K_\chi)$.

The induced module $\operatorname{ind}_\chi(K_\chi)$ has basis $v_i = x^i \otimes 1$, $0 \leq i < p$. The commutator formulas for the basis elements of $\mathfrak{g}$ show that $yx^i = x^i y - i\, x^{i-1} z$ in $U(\mathfrak{g})$ for all $i > 0$. Therefore

$y.v_0 = \chi(y)v_0$ and $z.v_0 = \chi(z)v_0$ imply for all i with $0 < i < p$

$$y.v_i = \chi(y)v_i - i\chi(z)v_{i-1}. \tag{1}$$

Every non-zero $U_\chi(\mathfrak{g})$–submodule of $\mathrm{ind}_\chi(K_\chi)$ contains a simple $U_\chi(\mathfrak{s})$–submodule, hence a non-zero vector v with $(y - \chi(y)).v = 0$. A look at (1) shows that the kernel of $y - \chi(y)$ on $\mathrm{ind}_\chi(K_\chi)$ is equal to Kv_0. (Here we use that $\chi(z) \neq 0$.) Therefore any non-zero $U_\chi(\mathfrak{g})$–submodule of $\mathrm{ind}_\chi(K_\chi)$ is equal to $\mathrm{ind}_\chi(K_\chi)$, this induced module is simple, and $E \simeq \mathrm{ind}_\chi(K_\chi)$.

Note that we get just one simple $U_\chi(\mathfrak{g})$–module for each χ as predicted by Theorem 3.3.

4.4. Definition: A Lie algebra $\mathfrak{g}$ is called *completely solvable* if there exists a chain of ideals in $\mathfrak{g}$

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_m = \mathfrak{g} \tag{1}$$

such that each $\mathfrak{g}_i$ has dimension i.

Remarks. a) Over **C** every (finite dimensional) solvable Lie algebra has a chain of ideals as in (1). This is no longer true in prime characteristic. For example, the semi-direct product of the Lie algebra from Example 1 with one of its simple p–dimensional modules (regarded as a commutative Lie algebra) is solvable, but not completely solvable.

b) If G is a solvable algebraic group, then $\mathrm{Lie}(G)$ is completely solvable. One gets in this case a chain as in (1) from a similar chain of closed connected normal subgroups in G^0 by taking the Lie algebras.

c) If a completely solvable Lie algebra $\mathfrak{g}$ is restricted then we can choose a chain as in (1) where each $\mathfrak{g}_i$ is a restricted ideal, i.e., stable under the pth power map. (To see this it suffices to find a one dimensional restricted ideal in $\mathfrak{g}$; one can then apply induction on the dimension to the factor algebra. The centre $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is a restricted ideal and the pth power map is semilinear on the centre. If $\mathfrak{z}(\mathfrak{g}) \neq 0$ then elementary arguments (cf. [**52**], Thm. II.3.6) yield an $x \in \mathfrak{z}(\mathfrak{g})$, $x \neq 0$ with $x^{[p]} = 0$ or $x^{[p]} = x$. Then Kx is the desired one dimensional restricted ideal. If the centre is 0, then we take any $x \in \mathfrak{g}$, $x \neq 0$ such that Kx is an ideal in $\mathfrak{g}$; such an x exists because $\mathfrak{g}$ is completely solvable. We have then $\mathrm{ad}(x)^2 = 0$, hence $0 = \mathrm{ad}(x)^p = \mathrm{ad}(x^{[p]})$ and therefore $x^{[p]} = 0$. So Kx is restricted.)

4.5. Theorem. *Suppose $\mathfrak{g}$ is a restricted, completely solvable Lie algebra. For each $\chi \in \mathfrak{g}^*$ every simple $U_\chi(\mathfrak{g})$–module is induced from a one dimensional module over some restricted Lie subalgebra of $\mathfrak{g}$.*

Remark. This theorem is proved in [**53**], Thm. 1b. It was later generalised to all restricted solvable $\mathfrak{g}$ (in case $p > 2$) in [**51**], Satz 3b, extending results for $\chi = 0$ in [**47**].

4.6. In order to complement Theorem 4.5 let us also describe a condition that tells us how to find restricted Lie subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ and one dimensional $U_\chi(\mathfrak{s})$–modules such that the induced $U_\chi(\mathfrak{g})$–module is simple.

This requires the notion of a polarisation. Let $f \in \mathfrak{g}^*$. Then f defines an alternating bilinear form

$$B_f : \mathfrak{g} \times \mathfrak{g} \to K, \qquad B_f(x, y) = f([x, y]). \tag{1}$$

The radical of this form is equal to the stabiliser of f under the coadjoint action:

$$\mathfrak{c}_\mathfrak{g}(f) = \{\, x \in \mathfrak{g} \mid x \cdot f = 0 \,\} = \{\, x \in \mathfrak{g} \mid f([x, y]) = 0 \text{ for all } y \in \mathfrak{g} \,\}. \tag{2}$$

This is a restricted Lie subalgebra of $\mathfrak{g}$. (Observe that $\mathfrak{g}^*$ under the coadjoint action is a restricted $\mathfrak{g}$–module.)

A *polarisation* of f is a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{p}$ is a maximal totally isotropic subspace in $\mathfrak{g}$ for B_f. So it is a Lie subalgebra $\mathfrak{p}$ with $f([\mathfrak{p}, \mathfrak{p}]) = 0$ and $\dim(\mathfrak{p}) = (\dim(\mathfrak{g}) +$

$\dim(\mathfrak{c}_{\mathfrak{g}}(f)))/2$. Each polarisation of f is automatically a restricted Lie subalgebra of $\mathfrak{g}$. (If $\mathfrak{p}$ is a polarisation of f and $x \in \mathfrak{p}$, then one checks easily that also $\mathfrak{p} + Kx^{[p]}$ is totally isotropic for B_f.)

Given a chain of ideals as in 4.4(1) we set

$$\mathfrak{s}_i = \{x \in \mathfrak{g}_i \mid f([x,y]) = 0 \text{ for all } y \in \mathfrak{g}_i\}$$

and $\mathfrak{p} = \mathfrak{s}_1 + \mathfrak{s}_2 + \cdots + \mathfrak{s}_m$. Then $\mathfrak{p}$ is a polarisation of f. (The characteristic 0 proof, e.g., in [**14**], 1.12.3 and 1.12.10, works also in our situation.) We shall call a polarisation constructed thus a *Vergne polarisation* of f. (This terminology is used in [**5**].) Using arguments similar to those in [**5**], 9.9, one can show:

PROPOSITION. *Let $f \in \mathfrak{g}^*$, let $\mathfrak{p}$ be a Vergne polarisation of f. Let $\chi \in \mathfrak{g}^*$ such that $\chi(x)^p = f(x)^p - f(x^{[p]})$ for all $x \in \mathfrak{p}$. Then $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} K_f$ is a simple $\mathfrak{g}$–module.*

Remarks. a) Here K_f denotes the one dimensional $\mathfrak{p}$–module where each $x \in \mathfrak{p}$ acts a multiplication by $f(x)$. There exists $\chi' \in \mathfrak{p}^*$ such that K_f is a $U_{\chi'}(\mathfrak{p})$–module. We can then take as χ any extension of χ' from $\mathfrak{p}$ to $\mathfrak{g}$.
b) The proposition does not extend to arbitrary polarisations: Take (e.g.) $\mathfrak{g}$ as in Example 1 and consider $f \in \mathfrak{g}^*$ with $f(x) = 1$ and $f(h) = 0$. Then $\mathfrak{p} = Kh$ is a polarisation of f. The $\mathfrak{p}$–module K_f is a $U_\chi(\mathfrak{p})$–module for any $\chi \in \mathfrak{g}^*$ with $\chi(h) = 0$. If we now choose χ with $\chi(x) = 0$, then $\text{ind}_\chi(K_f)$ is not simple. (This example is adapted from [**5**], 9.6.)
c) Theorem 4.5 can be made more precise: Given a simple $U_\chi(\mathfrak{g})$–module E, one can find $f \in \mathfrak{g}^*$ and a Vergne polarisation $\mathfrak{p}$ of f such that K_f is a $U_\chi(\mathfrak{p})$–module with $E \simeq \text{ind}_\chi(K_f)$.
d) If $\mathfrak{g}$ is a solvable, restricted Lie algebra over K that is not necessarily completely solvable, then one has to replace Vergne polarisations by more complicated constructions, see [**51**].
e) These results from the representation theory of Lie algebras in prime characteristic have been applied in characteristic 0 to the ideal theory of enveloping algebras. This was done by Mathieu in the proof of the bicontinuity of the Dixmier map in [**35**].

5. The Lie Algebra $\mathfrak{sl}_2(K)$

5.1. We are now getting ready to study Lie algebras of reductive algebraic groups. As a first example we look at the case $\mathfrak{g} = \mathfrak{sl}_2(K)$ with the usual basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These elements satisfy $e^{[p]} = 0$, $f^{[p]} = 0$, and $h^{[p]} = h$.

5.2. The simple $U_\chi(\mathfrak{g})$–modules for $\chi = 0$ have been known for a long time. The formulas in [**24**], Lemma 7.2 define for each integer $\lambda \geq 0$ a $\mathfrak{g}$–module structure on a vector space with basis $v_0, v_1, \ldots, v_\lambda$ (setting $v_{-1} = 0 = v_{\lambda+1}$). This module turns out to be simple for $0 \leq \lambda < p$, and one gets thus all simple $U_0(\mathfrak{g})$–modules.

5.3. The simple $U_\chi(\mathfrak{g})$–modules for $\chi \neq 0$ were first looked at by Block. He proves in [**3**], Lemma 5.1:

PROPOSITION. *If $p > 2$ and $\chi \neq 0$ then every simple $U_\chi(\mathfrak{sl}_2)$–module has dimension p.*

Block's lemma says more precisely that h acts on any such module diagonalisably with p distinct eigenvalues, each with multiplicity 1. His proof allows one to write down explicitly the action of e, f and h on a basis of the module. A precise classification of the simple modules was found later in [**46**].

5.4. We want to prove here a refined version that also tells us how many simple modules each $U_\chi(\mathfrak{g})$ has and that also works for $p=2$. We follow more or less Section 2 in [**19**].

Recall that the algebra $U_\chi(\mathfrak{sl}_2)$ depends (up to isomorphism) only on the orbit of χ under the automorphism group of $\mathfrak{sl}_2$ (as a restricted Lie algebra). The action of $\mathrm{GL}_2(K)$ on $\mathfrak{sl}_2$ by conjugation is an action by such automorphisms. Therefore $U_\chi(\mathfrak{sl}_2)$ depends only on the orbit of χ under $\mathrm{GL}_2(K)$.

Let us describe these orbits. Any $Y \in \mathfrak{gl}_2$ defines a linear form f_Y on $\mathfrak{sl}_2$ via $f_Y(X) = \mathrm{tr}(XY)$ where 'tr' denotes the trace. Since the bilinear form $(X,Y) \mapsto \mathrm{tr}(XY)$ is non-degenerate on $\mathfrak{gl}_2$ each linear form on $\mathfrak{sl}_2$ has the form f_Y with $Y \in \mathfrak{gl}_2$.

We get thus a surjective linear map $\mathfrak{gl}_2 \longrightarrow \mathfrak{sl}_2^*$, $Y \mapsto f_Y$. This map is clearly $\mathrm{GL}_2(K)$–equivariant. It therefore maps orbits to orbits. Each $\mathrm{GL}_2(K)$–orbit in $\mathfrak{gl}_2$ contains an element of the form

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}.$$

These elements are mapped to the following linear forms on $\mathfrak{sl}_2$

$$\begin{aligned} e &\longmapsto 0, & \qquad e &\longmapsto 0, \\ f &\longmapsto 0, & f &\longmapsto 1, \\ h &\longmapsto r-s, & h &\longmapsto 0. \end{aligned}$$

So each $\chi \in \mathfrak{sl}_2^*$ is conjugate to one of these forms. We call χ *semisimple* if it is conjugate to a form of the first type, and *nilpotent* if it is conjugate to a form of the second type or equal to 0. (One can check that only 0 is both semisimple and nilpotent, but that is not needed for the following arguments.)

We assume now that χ takes one of the forms above. Note that $\chi(e) = 0$ so $e^p = 0$ in $U_\chi(\mathfrak{sl}_2)$.

Let M be an irreducible $U_\chi(\mathfrak{sl}_2)$–module. Then $e^p.M = 0$. Thus $\{m \in M \mid e.m = 0\} \neq 0$. Moreover, this set is acted upon by h. Hence there exists $m_0 \in M$, $m_0 \neq 0$, such that $e.m_0 = 0$ and $h.m_0 = \lambda m_0$ for some $\lambda \in K$. Since $(h^p - h)_{|M} = \chi(h)^p{}_{|M}$ we have

$$\lambda^p - \lambda = \chi(h)^p.$$

Hence, for fixed χ, there are only p possibilities for λ.

Note that $Km_0 \subset M$ is a $U_\chi(Kh+Ke)$-submodule. By Frobenius reciprocity the induced module

$$Z_\chi(\lambda) \equiv \mathrm{ind}_\chi(Km_0) = U_\chi(\mathfrak{sl}_2) \otimes_{U_\chi(Kh+Ke)} Km_0$$

maps onto M:

$$Z_\chi(\lambda) \twoheadrightarrow M.$$

(The $U_\chi(\mathfrak{sl}_2)$–module $Z_\chi(\lambda)$ is an example of a 'baby Verma module' to be defined in general in the next section.) The set $\{v_i \equiv f^i \otimes m_0 \mid 0 \le i < p\}$ is a basis for $Z_\chi(\lambda)$ and we have the relations

$$\begin{aligned} h.v_i &= (\lambda - 2i)v_i \\ e.v_i &= \begin{cases} 0, & \text{if } i = 0, \\ i(\lambda - i + 1)v_{i-1}, & \text{if } i > 0, \end{cases} \\ f.v_i &= \begin{cases} v_{i+1}, & \text{if } i < p-1, \\ \chi(f)^p v_0, & \text{if } i = p-1. \end{cases} \end{aligned}$$

Case I : The form χ is non-zero and semisimple. We have then $\chi(h)^p \neq 0$ so $\lambda \notin \mathbf{F}_p$. It follows from the above relations that $\{v \in Z_\chi(\lambda) \mid e.v = 0\} = Kv_0$. Each non-zero submodule of $Z_\chi(\lambda)$ contains a non-zero vector annihilated by e. Therefore $Z_\chi(\lambda)$ is irreducible.

This shows that (in this case) each simple $U_\chi(\mathfrak{sl}_2)$–module is isomorphic to some $Z_\chi(\lambda)$. Conversely, each $\lambda \in K$ satisfying $\lambda^p - \lambda = \chi(h)^p$ defines first a one dimensional $U_\chi(Kh+Ke)$–module K_λ with h acting as λ, and e as 0. We get then the induced module $Z_\chi(\lambda) = \operatorname{ind}_\chi(K_\lambda)$ which is simple by the argument above. Furthermore, this simple module determines λ as the weight of h on the (one dimensional) subspace annihilated by e.

This shows that $U_\chi(\mathfrak{sl}_2)$ has precisely p non-isomorphic simple modules corresponding to the p distinct solutions of $\lambda^p - \lambda = \chi(h)^p$. All these simple modules have dimension p. Now 2.8(1) shows that $U_\chi(\mathfrak{sl}_2)$ is a semisimple ring isomorphic to a direct product of p copies of $M_p(K)$.

Case II : The form χ is non-zero and nilpotent. We have then $\chi(h) = 0$ and $\chi(f) = 1$. Assume that $p \neq 2$. Then the formulas above show that h acts on each v_i, $0 \leq i < p$ with a different eigenvalue. Therefore the eigenspaces of h in $Z_\chi(\lambda)$ are precisely the Kv_i. Now any non-zero $U_\chi(\mathfrak{sl}_2)$–submodule of $Z_\chi(\lambda)$ contains an eigenvector of h, hence one of the v_i. But then it also contains $v_0 = f^{p-i}.v_i$, hence is equal to $Z_\chi(\lambda)$. Therefore $Z_\chi(\lambda)$ is irreducible.

As in Case I each of the p solutions λ of $\lambda^p - \lambda = \chi(h)^p$ leads to such a simple module $Z_\chi(\lambda)$. However, now they are no longer pairwise non-isomorphic. Since $\chi(h) = 0$, the possible λ are precisely the elements of $\mathbf{F}_p$, which we identify with the integers $\{0, 1, \dots, p-1\}$. We see now

$$\{v \in Z_\chi(\lambda) \mid e.v = 0\} = \begin{cases} Kv_0 + Kv_{\lambda+1}, & \text{if } 0 \leq \lambda \leq p-2, \\ Kv_0, & \text{if } \lambda = p-1. \end{cases}$$

The line $Kv_{\lambda+1}$ in $Z_\chi(\lambda)$ is (for $\lambda \leq p-2$) a $U_\chi(Kh+Ke)$–submodule isomorphic to $K_{p-\lambda-2}$. We get thus by Frobenius reciprocity a non-zero homomorphism $Z_\chi(p-\lambda-2) \to Z_\chi(\lambda)$, which has to be an isomorphism, since both modules are simple. The description of the kernel of e on $Z_\chi(\lambda)$ shows that there cannot exist further isomorphisms.

It follows that $U_\chi(\mathfrak{sl}_2)$ has $(p+1)/2$ non-isomorphic simple modules, all of dimension p. Therefore $U_\chi(\mathfrak{sl}_2)$ cannot be semisimple in this case.

Case III : We have $\chi = 0$. In this case it is left to the reader to use the $Z_0(\lambda)$ to prove the claims on simple $U_0(\mathfrak{sl}_2)$–modules made at the beginning of this section.

5.5. We worked above with special representatives for the orbits of $\mathrm{GL}_2(K)$ in $\mathfrak{sl}_2^*$. However, it follows now for each non-zero semisimple χ that $U_\chi(\mathfrak{sl}_2)$ is semisimple and has p simple modules all of dimension p, while $U_\chi(\mathfrak{sl}_2)$ is not semisimple and has only $(p+1)/2$ simple modules (again all of dimension p) for each non-zero nilpotent χ (if $p > 2$). This implies of course that only $\chi = 0$ is both semisimple and nilpotent. (For $p = 2$ use the next subsection.)

5.6. Assume that $p = 2$ and consider χ as in Case II. We have $\lambda \in \{0, 1\}$ since $\chi(h) = 0$. If $\lambda = 0$, then both h and e annihilate $Z_\chi(\lambda)$. It follows that $Z_\chi(0)$ is a non-split extension of a simple one dimensional module L by itself where f acts as 1 on L, while e and h annihilate L. On the other hand, one can check that $Z_\chi(1)$ is simple.

5.7. If we take in 1.4 the case where $\mathfrak{g} = \mathfrak{sl}_2(\mathbf{C})$, then the simple $U_\varepsilon(\mathfrak{g})$–modules can be described very similarly to what has been done above. That was shown in Section 4 of [**9**].

6. Reductive Lie Algebras

6.1. We now turn to the main objects of our interest. Let G be a connected, reductive algebraic group over K and set $\mathfrak{g} = \mathrm{Lie}(G)$. We first set up some standard notation.

Let T be a maximal torus in G and set $\mathfrak{h} = \mathrm{Lie}(T)$. Let R be the root system of G. For each $\alpha \in R$ let $\mathfrak{g}_\alpha$ denote the corresponding root subspace of $\mathfrak{g}$. We choose a system R^+ of positive roots. Set $\mathfrak{n}^+$ equal to the sum of all $\mathfrak{g}_\alpha$ with $\alpha > 0$ and $\mathfrak{n}^-$ equal to the sum of all $\mathfrak{g}_\alpha$ with $\alpha < 0$. We have then the triangular decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$. These are Lie algebras of certain Borel subgroups of G containing T. The unipotent radicals of these Borel subgroups have Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ respectively. All of these subalgebras ($\mathfrak{h}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{b}^+$, $\mathfrak{b}^-$) are restricted subalgebras of $\mathfrak{g}$. Both $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are unipotent.

We choose for each root α a basis vector x_α for the (one dimensional) root subspace $\mathfrak{g}_\alpha$. (Occasionally we may want make a more specific choice of these x_α, but for the moment they can be arbitrary.) We have then $x_\alpha^{[p]} \in \mathfrak{g}_{p\alpha}$ since the adjoint action of T is compatible with the pth power map. Since $p\alpha$ is not a root this implies

$$x_\alpha^{[p]} = 0 \qquad \text{for all } \alpha \in R.$$

6.2. Since T is a direct product of multiplicative groups, its Lie algebra $\mathfrak{h}$ is a direct product of Lie algebras as in Example 2a. So $\mathfrak{h}$ is commutative and has a basis $h_1, h_2, \dots, h_r$ such that $h_i^{[p]} = h_i$ for all i. Each $\lambda \in \mathfrak{h}^*$ defines a one–dimensional $\mathfrak{h}$–module K_λ where each $h \in \mathfrak{h}$ acts as multiplication by $\lambda(h)$. Given $\chi \in \mathfrak{h}^*$ (or $\chi \in \mathfrak{g}^*$) then K_λ is a $U_\chi(\mathfrak{h})$–module if and only if $\lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p$ for all $h \in \mathfrak{h}$. The semilinearity of the map $h \mapsto h^p - h^{[p]}$ shows that it suffices to check these conditions for all $h = h_i$ where it takes the form $\lambda(h_i)^p - \lambda(h_i) = \chi(h_i)^p$. Set (for each such χ)

$$\begin{aligned}\Lambda_\chi &= \{\, \lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h} \,\}\\ &= \{\, \lambda \in \mathfrak{h}^* \mid \lambda(h_i)^p - \lambda(h_i) = \chi(h_i)^p \text{ for all } i \,\}.\end{aligned}$$

Our remarks above imply that K_λ is a $U_\chi(\mathfrak{h})$–module if and only if $\lambda \in \Lambda_\chi$. We see also that given χ there are precisely p possible values for $\lambda(h_i)$ if $\lambda \in \Lambda_\chi$. This shows that Λ_χ consists of p^r elements:

$$|\Lambda_\chi| = p^{\dim(\mathfrak{h})}.$$

If V is a $U_\chi(\mathfrak{h})$–module, then each h_i acts, by Example 2a, diagonalisably on V. Since $\mathfrak{h}$ is commutative, we can diagonalise the h_i simultaneously. This shows that V is a direct sum of *weight spaces* V_λ defined as

$$V_\lambda = \{\, v \in V \mid h.v = \lambda(h)\, v \text{ for all } h \in \mathfrak{h} \,\}.$$

It is clear that $V_\lambda = 0$ for $\lambda \notin \Lambda_\chi$.

6.3. The discussion of the special case $G = \mathrm{SL}_2(K)$ in the previous section shows that we have to expect special behaviour for small primes. In order to simplify our statements, we are basically going to ignore these small primes. This is achieved by making the following hypotheses:

(H1) The derived group $\mathcal{D}G$ of G is simply connected;
(H2) The prime p is good for $\mathfrak{g}$;
(H3) There exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$.

However, we will occasionally insert a remark stating which of these hypotheses really are needed or whether it is unknown what exactly is required for a certain result to hold.

6.4. Let us see first what these hypotheses amount to.
a) (H2): There is an 'abstract' definition of what it means for a prime to be good for $\mathfrak{g}$. But it is quicker to give an explicit description. A prime is good for $\mathfrak{g}$ if it is good for all irreducible components of the root system R. And the 'bad' (i.e., not good) primes for an irreducible root system are:

- none for type A_n $(n \geq 1)$,
- 2 for types B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$,
- 2 and 3 for types E_6, E_7, F_4, and G_2,
- 2, 3, and 5 for type E_8.

b) (H3): This hypothesis ensures the existence of a G–module isomorphism

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*.$$

If G is almost simple and simply connected, then we have such an isomorphism whenever $\mathfrak{g}$ is simple as a Lie algebra. That holds if and only if:

- for type A_n $(n \geq 1)$ if p does not divide $n+1$,
- for types B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, F_4, and E_7 if $p \neq 2$,
- for types G_2 and E_6 if $p \neq 3$,
- for type E_8 always.

This shows that here (for almost simple G) usually (H2) implies (H3). The one (big) exception is G of type A_n, i.e., $G = \mathrm{SL}_{n+1}(K)$. Note on the other hand that $G = \mathrm{GL}_{n+1}(K)$ always satisfies (H3), since the trace form on $M_n(K)$ is always non-degenerate.
c) (H1): Again we look first at almost simple G. Then G has a finite covering $\widetilde{G} \to G$ with $\widetilde{G}$ simply connected. This induces a Lie algebra homomorphism $\mathrm{Lie}(\widetilde{G}) \to \mathrm{Lie}(G) = \mathfrak{g}$. If $\mathrm{Lie}(\widetilde{G})$ is simple, then this homomorphism is injective, hence an isomorphism (because both $\mathrm{Lie}(\widetilde{G})$ and $\mathrm{Lie}(G)$ have dimension equal to $\dim(G) = \dim(\widetilde{G})$). So, if we are just interested in the representation theory of $\mathfrak{g}$, we just replace G by $\widetilde{G}$ in this case. Note that the table under b) tells us when $\mathrm{Lie}(\widetilde{G})$ is simple.

For arbitrary G there is always a finite covering $\widetilde{G} \to G$ such that $\mathcal{D}\widetilde{G}$ is simply connected. But it seems to be more difficult to see in general when $\mathrm{Lie}(\widetilde{G})$ and $\mathfrak{g}$ can be identified.

6.5. It is not difficult to show that if G satisfies (H1)–(H3) then so too does any Levi factor in G. This is important for inductive constructions to work.

6.6. Assume from now on that G satisfies (H1)–(H3). So we have by (H3) a non-degenerate G–invariant bilinear form $(\, , \,)$ on $\mathfrak{g}$. The T–invariance of this form implies easily that each $\mathfrak{g}_\alpha$ with $\alpha \in R \cup \{0\}$ (where $\mathfrak{g}_0 = \mathfrak{h}$) is orthogonal to each $\mathfrak{g}_\beta$ with $\beta \neq -\alpha$. If follows that $(\, , \,)$ induces an isomorphism $\mathfrak{g}_\alpha \xrightarrow{\sim} (\mathfrak{g}_{-\alpha})^*$ and that $(\, , \,)$ is non-degenerate on $\mathfrak{h}$. [Using this it is easy to show that every Levi factor in G satisfies (H3).]

LEMMA. *Each $\chi \in \mathfrak{g}^*$ is conjugate under G to an element $\chi' \in \mathfrak{g}^*$ with $\chi'(\mathfrak{n}^+) = 0$.*

Proof. There exists $y \in \mathfrak{g}$ such that $\chi(x) = (y, x)$ for all x. Any element in $\mathfrak{g}$ is conjugate under the adjoint action of G to an element in $\mathfrak{b}^+$, see [**4**], Prop. 14.25. So let $y' \in G.y \cap \mathfrak{b}^+$ and define $\chi' \in \mathfrak{g}^*$ by $\chi'(x) = (y', x)$. Then $y' \in G.y$ implies $\chi' \in G.\chi$ while $y' \in \mathfrak{b}^+$ yields $\chi'(\mathfrak{n}^+) = 0$ by the orthogonality statements above.
Remark. We used here only one of our hypotheses, (H3). In [**33**], Lemma 3.2, Kac and Weisfeiler show that the lemma holds for each almost simple G except perhaps for $G =$

SO_{2n+1}, $n \geq 1$ in characteristic 2. Their arguments can be used to prove the lemma for all G satisfying (H1).

6.7. Recall that the algebra $U_\chi(\mathfrak{g})$ depends, up to isomorphism, only on the G-orbit of $\chi \in \mathfrak{g}^*$. Therefore the lemma tells us that it suffices to look at χ with $\chi(\mathfrak{n}^+) = 0$.

So assume that $\chi(\mathfrak{n}^+) = 0$. Since $\mathfrak{n}^+$ is unipotent, by Proposition 3.2, the only simple $U_\chi(\mathfrak{n}^+)$–module is K. Since $\mathfrak{n}^+$ is an ideal of $\mathfrak{b}^+$ and the Lie algebra $\mathfrak{b}^+/\mathfrak{n}^+ \simeq \mathfrak{h}$ is Abelian we see that the simple $U_\chi(\mathfrak{b}^+)$–modules are the K_λ with $\lambda \in \Lambda_\chi$.

Let M be a simple $U_\chi(\mathfrak{g})$–module. Then M contains a simple $U_\chi(\mathfrak{b}^+)$–module, say K_λ. Hence, by Frobenius reciprocity, we have a non-zero homomorphism

$$Z_\chi(\lambda) \equiv U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}^+)} K_\lambda = \mathrm{ind}_\chi(K_\lambda) \twoheadrightarrow M.$$

This shows:

Proposition. *Suppose $\chi(\mathfrak{n}^+) = 0$. Then each simple $U_\chi(\mathfrak{g})$–module is the homomorphic image of some $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$.*

Remark. This proposition is basically contained in [**45**], cf. Prop. 2 and the proof of Thm. 1. (Rudakov works with $\mathfrak{n}^-$ instead of $\mathfrak{n}^+$.) The proposition is then stated explicitly in [**19**], 1.5, and (for $\mathfrak{g} = \mathfrak{sl}_n$) in [**37**], Thm. 1.

6.8. Definition: Any module $Z_\chi(\lambda)$ is called a *baby Verma module*. We shall use the notation v_λ for the 'standard generator' $v_\lambda = 1 \otimes 1$ of $Z_\chi(\lambda)$. Then the set $\{\prod_{\alpha \in R^+} x_{-\alpha}^{a(\alpha)}.v_\lambda \mid 0 \leq a(\alpha) < p\}$ is a basis of $Z_\chi(\lambda)$. We have an isomorphism of $\mathfrak{n}^-$–modules

$$U_\chi(\mathfrak{n}^-) \xrightarrow{\sim} Z_\chi(\lambda).$$

6.9. The name 'baby Verma module' was first applied to the $Z_\chi(\lambda)$ in the case $\chi = 0$. These objects are clearly constructed analogously to the Verma modules for complex semisimple Lie algebras, but much smaller. But one should be warned that they have some quite different properties, in particular for $\chi \neq 0$.

A baby Verma module can have more than one maximal submodule (in contrast to the characteristic 0 situation). Furthermore, it is possible for $Z_\chi(\lambda) \simeq Z_\chi(\lambda')$ while $\lambda \neq \lambda'$. The problem is that the usual arguments over $\mathbf{C}$ cannot be applied in our situation, since the weights (contained in $\mathfrak{h}^*$) have no ordering.

We have already seen an example for the second phenomenon: We got for $\mathfrak{g} = \mathfrak{sl}_2$ in Case II that $Z_\chi(\lambda) \simeq Z_\chi(p - \lambda - 2)$ if $0 \leq \lambda \leq p - 2$.

This $\mathfrak{sl}_2$–example can be generalised to arbitrary $\mathfrak{g}$ as follows: Assume $\chi(\mathfrak{b}^+) = 0$ and suppose that α is a simple root such that $\chi(x_{-\alpha}) \neq 0$. The Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2$ as a restricted Lie algebra. (This holds automatically for $p > 2$ while it follows from (H1) in case $p = 2$.) We can therefore assume that x_α and $x_{-\alpha}$ have been chosen such that $h_\alpha = [x_\alpha, x_{-\alpha}]$ satisfies $[h_\alpha, x_\alpha] = 2x_\alpha$ and $[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$ and $h_\alpha^{[p]} = h_\alpha$. Let $\lambda \in \Lambda_\chi$. We have $\lambda(h_\alpha)^p - \lambda(h_\alpha) = \chi(h_\alpha)^p = 0$. So there is an integer a with $0 \leq a < p$ and $\lambda(h_\alpha) = a.1$. Standard calculations show that $\mathfrak{n}^+.(x_{-\alpha}^{a+1}.v_\lambda) = 0$ and that $\mathfrak{h}$ acts on $x_{-\alpha}^{a+1}.v_\lambda$ via $\lambda - (a+1)\alpha$. (We write here α instead of its derivative $d\alpha$ by abuse of notation.) Thus we get a homomorphism

$$Z_\chi(\lambda - (a+1)\alpha) \longrightarrow Z_\chi(\lambda)$$
$$v_{\lambda-(a+1)\alpha} \longmapsto x_{-\alpha}^{a+1}.v_\lambda,$$

This map is surjective since its image contains

$$x_{-\alpha}^{p-(a+1)}.(x_{-\alpha}^{a+1}.v_\lambda) = x_{-\alpha}^p.v_\lambda = \chi(x_{-\alpha})^p v_\lambda = cv_\lambda$$

(for some $c \neq 0$). Since both baby Verma modules have the same dimension (equal to $p^{\dim(\mathfrak{n}^-)}$) this map has to be an isomorphism

$$Z_\chi(\lambda - (a+1)\alpha) \xrightarrow{\sim} Z_\chi(\lambda). \tag{1}$$

If $a \leq p-2$, then $\lambda - (a+1)\alpha \neq \lambda$.

Let me now describe (without proof) an example of a baby Verma module with two maximal submodules. Take $\mathfrak{g} = \mathfrak{sl}_3$ and denote the simple roots by α and β. Assume that x_α and $x_{-\alpha}$ (and similarly x_β and $x_{-\beta}$) have been normalised as above. The elements $h_\alpha = [x_\alpha, x_{-\alpha}]$ and $h_\beta = [x_\beta, x_{-\beta}]$ are a basis of $\mathfrak{h}$. Choose a linear form χ vanishing on $\mathfrak{h}$ and on every $\mathfrak{g}_\gamma$ except $\gamma = -\alpha-\beta$. Let a and b be non-negative integers with $a+b \leq p-2$. Define $\lambda \in \mathfrak{h}^*$ by $\lambda(h_\alpha) = a.1$ and $\lambda(h_\beta) = b.1$. We have $\lambda \in \Lambda_\chi$ since $\chi(\mathfrak{h}) = 0$. We get as above homomorphisms

$$\begin{array}{ccc} Z_\chi(\lambda-(a+1)\alpha) \longrightarrow Z_\chi(\lambda) & \text{and} & Z_\chi(\lambda-(b+1)\beta) \longrightarrow Z_\chi(\lambda) \\ v_{\lambda-(a+1)\alpha} \longmapsto x_{-\alpha}^{a+1}.v_\lambda & & v_{\lambda-(b+1)\beta} \longmapsto x_{-\beta}^{b+1}.v_\lambda. \end{array}$$

These maps are no longer isomorphisms. We have $\chi(x_{-\alpha}) = \chi(x_{-\beta}) = 0$, hence $x_{-\alpha}^p = 0 = x_{-\beta}^p$ in $U_\chi(\mathfrak{g})$. Using this one checks that the images of our homomorphisms are submodules of codimension $(a+1)p^2$ and $(b+1)p^2$, respectively, in $Z_\chi(\lambda)$. These submodules turn out to be maximal and distinct. A reasonable proof of this fact requires more information about simple $U_\chi(\mathfrak{g})$–modules than we have seen so far. However, the reader may find it not too hard to check for $p = 2$ and $a = b = 0$ that $Z_\chi(\lambda)$ is the direct sum of these two submodules (both of dimension 4 in this case). One gets such a direct sum decomposition always when $a+b+2 = p$.

6.10. In the quantum situation from 1.4 there is a result analogous to Lemma 6.6 in [**11**], Theorem 6.1(a). The corollary to that theorem is then an analogue to Proposition 6.7.

7. Premet's Theorem and Applications

Keep all assumptions and notations from the previous section. We want to state in this section a theorem proved by Premet in [**42**]. We then illustrate the power of this result by several applications. The proof of the theorem itself will then be discussed in the next section.

7.1. Let $\chi \in \mathfrak{g}^*$ and let $\mathfrak{c}_\mathfrak{g}(\chi) = \{x \in \mathfrak{g} \mid \chi([x,\mathfrak{g}]) = 0\}$ be the centraliser of χ in $\mathfrak{g}$. If χ is the image of $y \in \mathfrak{g}$ under an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, then $\mathfrak{c}_\mathfrak{g}(\chi)$ is equal to the usual centraliser $\mathfrak{c}_\mathfrak{g}(y) = \{x \in \mathfrak{g} \mid [x,y] = 0\}$ of y in $\mathfrak{g}$.

Theorem. *Let $\mathfrak{m}$ be a restricted Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \cap \mathfrak{c}_\mathfrak{g}(\chi) = 0$. Then each $U_\chi(\mathfrak{g})$–module is projective over $U_\chi(\mathfrak{m})$.*

7.2. For applications the following corollary will be more convenient:

Corollary. *Let $\mathfrak{m}$ be a unipotent restricted subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \cap \mathfrak{c}_\mathfrak{g}(\chi) = 0$. If $\chi([\mathfrak{m},\mathfrak{m}]) = 0$ and $\chi(\mathfrak{m}^{[p]}) = 0$, then each $U_\chi(\mathfrak{g})$–module is free over $U_\chi(\mathfrak{m})$.*

Proof. The assumption $\chi([\mathfrak{m},\mathfrak{m}]) = 0$ implies that χ defines a one dimensional $\mathfrak{m}$–module K_χ where each $x \in \mathfrak{m}$ acts as multiplication by $\chi(x)$. The assumption $\chi(\mathfrak{m}^{[p]}) = 0$ implies that each $x^p - x^{[p]} - \chi(x)^p$ annihilates K_χ. Therefore K_χ is actually a $U_\chi(\mathfrak{m})$–module. Now the claim follows from Corollary 3.4 and the Theorem.

Remark. Theorem 7.1 and hence Corollary 7.2 are proved for semisimple G satisfying (H1) in [**42**] excluding only $p = 2$ for types B, C, and F_4 and $p = 3$ for type G_2. These restrictions were removed in [**44**] where at the same time a mistake in [**42**] is taken care of.

In the reductive case the theorem and its corollary are proved for G satisfying (H1) and (H2) in [**43**], 4.3.

7.3. An element in $\mathfrak{g}^*$ is called *semisimple* if it is the image under $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ of a semisimple element in $\mathfrak{g}$. An element in $\mathfrak{g}$ is semisimple if and only if it is conjugate to an element in $\mathfrak{h}$. Therefore an element in $\mathfrak{g}^*$ is semisimple if and only if it is conjugate to a linear form $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = \chi(\mathfrak{n}^-) = 0$.

A semisimple element in $\mathfrak{g}$ is called 'regular semisimple' if its centraliser has dimension equal to $\dim(\mathfrak{h})$. Any $h \in \mathfrak{h}$ is regular if and only if its centraliser is equal to $\mathfrak{h}$. An element in $\mathfrak{g}^*$ is called *regular semisimple* if it is the image under $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ of a regular semisimple element in $\mathfrak{g}$. It follows that a regular semisimple element in $\mathfrak{g}^*$ is conjugate to some $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^-) = 0 = \chi(\mathfrak{n}^+)$ and $\mathfrak{c}_\mathfrak{g}(\chi) = \mathfrak{h}$. (The last condition can be checked to be equivalent to $\chi([x_\alpha, x_{-\alpha}]) \neq 0$ for all $\alpha \in R$.)

PROPOSITION. a) *Let $\chi \in \mathfrak{g}^*$ be regular semisimple. Then $U_\chi(\mathfrak{g})$ is a semisimple algebra, isomorphic to a direct product of $p^{\dim \mathfrak{h}}$ matrix algebras over K of dimension $(p^{\dim \mathfrak{n}^-})^2$.*
b) *Suppose $\chi \in \mathfrak{g}^*$ satisfies $\chi(\mathfrak{n}^-) = 0 = \chi(\mathfrak{n}^+)$ and $\mathfrak{c}_\mathfrak{g}(\chi) = \mathfrak{h}$. Then each $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$ is simple. Each simple $U_\chi(\mathfrak{g})$–module is isomorphic to exactly one $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$.*

Proof. Consider χ as in b). We have $\mathfrak{n}^- \cap \mathfrak{c}_\mathfrak{g}(\chi) = 0$. Since $\mathfrak{n}^-$ is unipotent with $\chi(\mathfrak{n}^-) = 0$, Corollary 7.2 implies each $U_\chi(\mathfrak{g})$–module is free over $U_\chi(\mathfrak{n}^-)$.

Recall that $Z_\chi(\lambda) \simeq U_\chi(\mathfrak{n}^-)$ as $\mathfrak{n}^-$–modules for any $\lambda \in \Lambda_\chi$. Therefore each proper non-zero submodule of $Z_\chi(\lambda)$ is not free over $U_\chi(\mathfrak{n}^-)$. It follows that $Z_\chi(\lambda)$ is a simple $U_\chi(\mathfrak{g})$–module.

Now Proposition 6.7 shows that each simple $U_\chi(\mathfrak{g})$–module is isomorphic to some $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$. Furthermore, in this case λ is determined by $Z_\chi(\lambda)$ as it is the weight of $\mathfrak{h}$ on $Z_\chi(\lambda)/\mathfrak{n}^- Z_\chi(\lambda)$. Thus there are $p^{\dim \mathfrak{h}}$ simple $U_\chi(\mathfrak{g})$–modules of dimension $p^{\dim \mathfrak{n}^-}$. Since $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}} = p^{\dim \mathfrak{h} + 2 \dim \mathfrak{n}^-}$ this implies $U_\chi(\mathfrak{g})$ is semisimple.

It follows now for each regular semisimple χ that $U_\chi(\mathfrak{g})$ is semisimple and has $p^{\dim \mathfrak{h}}$ simple modules of dimension $p^{\dim \mathfrak{n}^-}$. This proves b) as well as a) for χ as in b). The general case in a) follows now using 2.9.
Remarks. a) Note that this generalises Case I for $\mathfrak{sl}_2$.
b) The classification of the simple $U_\chi(\mathfrak{g})$–modules in this case is due to Rudakov, see [**45**], Prop. 3. He also proved the irreducibility of the $Z_\chi(\lambda)$, see [**45**], Thm. 3. (His assumptions on $\mathfrak{g}$ are somewhat more restrictive than those here.)

7.4. Let us return to arbitrary χ. There is the Jordan decomposition in $\mathfrak{g}$: Each $y \in \mathfrak{g}$ can be written uniquely $y = y_s + y_n$ with y_s semisimple, y_n nilpotent, and $[y_s, y_n] = 0$. We use now our isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ to get a Jordan decomposition in $\mathfrak{g}^*$: If $\chi \in \mathfrak{g}^*$ is the image of $y \in \mathfrak{g}$, then we decompose $y = y_s + y_n$ as above, let χ_s and χ_n denote the images of y_s and y_n, respectively, and call $\chi = \chi_s + \chi_n$ the *Jordan decomposition* of χ. We say that χ is *nilpotent* if $\chi = \chi_n$. A comparison with the previous subsection shows that χ is semisimple if and only if $\chi = \chi_s$.

The Jordan decomposition in $\mathfrak{g}$ has the property that $\mathfrak{c}_\mathfrak{g}(y) = \mathfrak{c}_\mathfrak{g}(y_s) \cap \mathfrak{c}_\mathfrak{g}(y_n)$. This implies that $\mathfrak{c}_\mathfrak{g}(\chi) = \mathfrak{c}_\mathfrak{g}(\chi_s) \cap \mathfrak{c}_\mathfrak{g}(\chi_n)$.

Recall that we assume p to be good for $\mathfrak{g}$ by (H2). This implies that $\mathfrak{l} = \mathfrak{c}_\mathfrak{g}(\chi_s) = \mathfrak{c}_\mathfrak{g}(y_s)$ is a Levi subalgebra of some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. (One can assume by conjugating that $y_s \in \mathfrak{h}$. Then $\mathfrak{c}_\mathfrak{g}(y_s)$ is the direct sum of $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $(d\alpha)(y_s) = 0$. Using the goodness of p one can check that the set of these α is conjugate under the Weyl group to a set of the form $R_I = R \cap \mathbf{Z}I$ for some subset I of the basis of our root system. Then $\mathfrak{c}_\mathfrak{g}(y_s)$ is conjugate to the Lie subalgebra $\mathfrak{g}_I$ defined as the direct sum of $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $\alpha \in R_I$. Now $\mathfrak{g}_I$ is a Levi factor in the parabolic subalgebra $\mathfrak{p}_I = \mathfrak{g}_I + \mathfrak{b}^+$.)

So there is a parabolic subgroup P of G with $\mathfrak{p} = \mathrm{Lie}(P)$. Let $\mathfrak{u}$ denote the Lie algebra of the unipotent radical of P. Then $\mathfrak{u}$ is a unipotent restricted Lie subalgebra of $\mathfrak{g}$ and an ideal in $\mathfrak{p}$; we have $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. We have $\mathfrak{u} = \mathfrak{p}^\perp$ with respect to our invariant form $(\ ,\ )$. (It is enough to check this for the standard parabolic subalgebras $\mathfrak{p}_I$ where it follows from $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ for $\alpha + \beta \neq 0$.)

We have now $\mathfrak{u} \cap \mathfrak{c}_\mathfrak{g}(\chi) = 0$ since $\mathfrak{c}_\mathfrak{g}(\chi) \subset \mathfrak{c}_\mathfrak{g}(\chi_s) = \mathfrak{l}$. Furthermore $\chi(\mathfrak{u}) = 0$ since $\chi(x) = (y, x)$ and $y \in \mathfrak{c}_\mathfrak{g}(y_s) = \mathfrak{l}$ is orthogonal to $\mathfrak{u}$. Hence, by Corollary 7.2, any $U_\chi(\mathfrak{g})$–module is free over $U_\chi(\mathfrak{u})$.

For each $\mathfrak{u}$–module M set $M^\mathfrak{u} = \{ m \in M \mid xm = 0 \text{ for all } x \in \mathfrak{u} \}$. If M is a $\mathfrak{g}$–module then $M^\mathfrak{u}$ is an $\mathfrak{l}$–submodule of M because $\mathfrak{l}$ normalises $\mathfrak{u}$. We get thus a functor

$$\{ U_\chi(\mathfrak{g})\text{–modules} \} \longrightarrow \{ U_\chi(\mathfrak{l})\text{–modules} \}, \qquad M \longmapsto M^\mathfrak{u}. \tag{1}$$

There is a functor in the other direction:

$$\{ U_\chi(\mathfrak{l})\text{–modules} \} \longrightarrow \{ U_\chi(\mathfrak{g})\text{–modules} \}, \qquad V \longmapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V \tag{2}$$

regarding any $U_\chi(\mathfrak{l})$–module V as a $\mathfrak{p}$–module with $\mathfrak{u}$ acting trivially. (Since $\chi(\mathfrak{u}) = 0$ this extension from $\mathfrak{l}$ to $\mathfrak{p}$ yields a $U_\chi(\mathfrak{p})$–module.)

Proposition. *The functors $V \mapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V$ and $M \mapsto M^\mathfrak{u}$ are inverse equivalences of categories. They induce a bijection between isomorphism classes of simple modules.*

Proof. By Proposition 3.2 and Corollary 3.4 the restricted enveloping algebra $U_0(\mathfrak{u})$ is an indecomposable $\mathfrak{u}$–module. Since each restricted enveloping algebra is a Frobenius algebra (by a theorem of Berkson in [**2**]) it follows that $U_0(\mathfrak{u})$ has a simple socle, hence that

$$\dim U_0(\mathfrak{u})^\mathfrak{u} = 1. \tag{3}$$

Since each $U_\chi(\mathfrak{g})$–module M is free over $U_0(\mathfrak{u})$ this implies

$$\dim M = p^{\dim \mathfrak{u}} \dim M^\mathfrak{u}. \tag{4}$$

Consider a $U_\chi(\mathfrak{l})$–module V. We have $\dim(\mathfrak{g}/\mathfrak{p}) = \dim(\mathfrak{u})$ (since $\mathfrak{p}^\perp = \mathfrak{u}$), hence

$$\dim U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V = p^{\dim \mathfrak{u}} \dim V. \tag{5}$$

It is clear that $1 \otimes V$ is contained in $(U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V)^\mathfrak{u}$. Therefore (4) and (5) show that $v \mapsto 1 \otimes v$ is an isomorphism

$$V \xrightarrow{\sim} (U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V)^\mathfrak{u}.$$

On the other hand Frobenius reciprocity yields for each $U_\chi(\mathfrak{g})$–module M a homomorphism

$$U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} M^\mathfrak{u} \longrightarrow M$$

given by $u \otimes m \mapsto u.m$. Both modules have the same dimension. This implies then first for simple M and then (by induction on the length) for all M that this map is an isomorphism. (Note that $M \mapsto M^\mathfrak{u}$ is exact because $U_\chi(\mathfrak{g})$–modules are free over $U_\chi(\mathfrak{u})$.)

Remarks. a) This result goes back to Weisfeiler and Kac who proved in [**53**], Thm. 2 that each simple $U_\chi(\mathfrak{g})$–module is isomorphic to some $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V$. The more precise statement here is due to [**19**], Thm. 3.2 and Thm. 8.5.

b) By 6.5 also $\mathfrak{l}$ satisfies hypotheses (H1)–(H3). So this proposition reduces the study of general χ to the case where χ is nilpotent.

c) Kac and Weisfeiler showed in [**33**] that one can construct (in most cases) a Jordan decomposition in $\mathfrak{g}^*$ even if there is not an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. They assume G to be almost simple excluding $G = \mathrm{SO}_{2n+1}(K)$ if $p = 2$. Their arguments can be applied to all G satisfying (H1).

Using this one can prove Proposition 7.4 without the assumption (H3). The hypothesis (H2) is used only to ensure that the centraliser of χ_s (the semisimple part of $\chi \in \mathfrak{h}^*$) is a Levi factor $\mathfrak{l}$ in some parabolic subalgebra. One can check that the proof works equally well when we just assume that the centraliser of χ_s is contained in $\mathfrak{l}$. Therefore Proposition 7.4 yields [not assuming (H2)] a reduction to the case where the centraliser of χ_s is not contained in any proper Levi subalgebra of $\mathfrak{g}$.

7.5. In the quantum situation from 1.4 one has a similar reduction result proved in [**10**]. However, except for type A_n the reduction is not quite as complete as in (good) characteristic since there are semi-simple elements different from 1 whose centraliser is not contained in a proper parabolic subgroup.

7.6. The following result was conjectured in [**53**] and became known as the *Kac-Weisfeiler conjecture*. It was proved by Premet in [**40**].

Proposition. *Let $\chi \in \mathfrak{g}^*$ and M be a $U_\chi(\mathfrak{g})$–module. Then*

$$p^{\dim(G.\chi)/2} \mid \dim M.$$

Using Proposition 7.4 one can reduce the proof of this proposition to the case where χ is nilpotent. So let $e \in \mathfrak{g}$ be a nilpotent element and consider $\chi \in \mathfrak{g}^*$ with $\chi(x) = (e, x)$. Note that the orbit $G.e$ is isomorphic to the orbit $G.\chi$; we have in particular $\dim(G.\chi) = \dim(G.e)$.

It is clear what we need in order to prove this proposition using Corollary 7.2: We want a unipotent restricted Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ with $\chi([\mathfrak{m},\mathfrak{m}]) = 0 = \chi(\mathfrak{m}^{[p]})$, with $\mathfrak{m} \cap \mathfrak{c}_\mathfrak{g}(\chi) = 0$ and $\dim \mathfrak{m} = \dim(G.\chi)/2$. If so then M is free over $U_\chi(\mathfrak{m})$ and $\dim U_\chi(\mathfrak{m}) = p^{\dim \mathfrak{m}}$ yields the claim.

7.7. Let me first describe how to find $\mathfrak{m}$ as above over $\mathbf{C}$. So assume for the moment that G is a reductive algebraic group over $\mathbf{C}$ and that $\mathfrak{g} = \mathrm{Lie}(G)$. Suppose that $(\ ,\ )$ is a G-invariant non-degenerate bilinear form on $\mathfrak{g}$ and that $e \in \mathfrak{g}$ is nilpotent. Then we want to construct a Lie subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{m}$ consists of nilpotent elements (that replaces the condition 'unipotent') and such that $\mathfrak{m}$ satisfies $(e, [\mathfrak{m},\mathfrak{m}]) = 0$ and $\mathfrak{m} \cap \mathfrak{c}_\mathfrak{g}(e) = 0$ as well as $\dim \mathfrak{m} = \dim(G.e)/2$. (We do not have a characteristic 0 analogue to $0 = \chi(\mathfrak{m}^{[p]})$.)

The Jacobson-Morozov theorem says that there are $f, h \in \mathfrak{g}$ such that (e, f, h) is an $\mathfrak{sl}_2$–triple. This means that $[e, f] = h$ and $[h, e] = 2e$ and $[h, f] = -2f$. So $\mathbf{C}e + \mathbf{C}f + \mathbf{C}h$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. We can decompose $\mathfrak{g}$ under the adjoint action of this subalgebra. We get in particular that

$$\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}} \mathfrak{g}(i) \qquad \text{where} \qquad \mathfrak{g}(i) = \{\, x \in \mathfrak{g} \mid [h, x] = ix \,\}.$$

The centraliser of e in $\mathfrak{g}$ is now the span of the highest weight vectors in the distinct simple $\mathfrak{sl}_2(\mathbf{C})$–submodules. This implies that

$$\mathfrak{c}_\mathfrak{g}(e) \;\subset\; \bigoplus_{i\geq 0} \mathfrak{g}(i) \tag{1}$$

and

$$\dim \mathfrak{c}_\mathfrak{g}(e) = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1) \tag{2}$$

see [**24**], 7.2. In characteristic 0 the centraliser $\mathfrak{c}_\mathfrak{g}(e)$ of e in $\mathfrak{g}$ is the Lie algebra of the centraliser $C_G(e)$ of e in G. This implies that

$$\dim G.e = \dim G - \dim C_G(e) = \dim \mathfrak{g} - \dim \mathfrak{g}(0) - \dim \mathfrak{g}(1).$$

The decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ is a grading of $\mathfrak{g}$ as a Lie algebra, that is, it satisfies $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ for all i and j. We have clearly $e \in \mathfrak{g}(2)$. The G–invariance of $(\ ,\ )$ implies that $\mathfrak{g}(i) \perp \mathfrak{g}(j)$ if $i+j \neq 0$. Since $(\ ,\ )$ is non-degenerate it induces therefore a non-degenerate pairing between $\mathfrak{g}(i)$ and $\mathfrak{g}(-i)$ (for each i). This implies in particular that $\dim \mathfrak{g}(i) = \dim \mathfrak{g}(-i)$. (That could also be deduced from $\mathfrak{sl}_2$ representation theory.) Our earlier dimension formula yields now:

$$\frac{1}{2} \dim(G.e) = \sum_{i \geq 2} \dim(\mathfrak{g}(-i)) + \frac{1}{2} \dim(\mathfrak{g}(-1)). \tag{3}$$

On $\mathfrak{g}(-1)$ there is a symplectic bilinear form, $\langle\ ,\ \rangle$, given by

$$\langle x, y \rangle = (e, [x, y]) = \chi([x, y]). \tag{4}$$

This form is non-degenerate: Take $x \in \mathfrak{g}(-1)$, $x \neq 0$. Then $[e, x] \neq 0$ by (1) and $[e, x] \in \mathfrak{g}(1)$. The non-degeneracy of $(\ ,\ )$ yields therefore $y \in \mathfrak{g}(-1)$ with $0 \neq ([e, x], y) = (e, [x, y]) = \langle x, y \rangle$ using the invariance of $(\ ,\ )$ for the first equality.

Take $\mathfrak{g}(-1)' \subset \mathfrak{g}(-1)$ to be a maximal isotropic subspace with respect to $\langle\ ,\ \rangle$. It satisfies $\dim \mathfrak{g}(-1)' = (\dim \mathfrak{g}(-1))/2$. Set $\mathfrak{m} = \bigoplus_{i \geq 2} \mathfrak{g}(-i) \oplus \mathfrak{g}(-1)'$. Then $\dim \mathfrak{m} = \dim(G.e)/2$ and $\mathfrak{m} \cap \mathfrak{c}_{\mathfrak{g}}(e) = 0$ and $(e, [\mathfrak{m}, \mathfrak{m}]) = 0$ follow from the construction and the formulas above. The basic properties of a grading show that $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$ consisting of nilpotent elements. (Note that $\mathfrak{g}(-i) = 0$ for $i \gg 0$.) So $\mathfrak{m}$ satisfies our requirements.

7.8. Let us return to characteristic p and our usual set-up. One can argue more or less as in 7.7 if p is large with respect to the root system. (It will do to assume p greater than 3 times the maximum of the Coxeter numbers of the irreducible components of R.) In this case the Jacobson-Morozov theorem holds in $\mathfrak{g}$ ([**6**], 5.3.2 and 5.5.2), then $\mathfrak{g}$ is semisimple as a module over the corresponding $\mathfrak{sl}_2$ ([**6**], 5.4.8), and one gets a grading of $\mathfrak{g}$ that has the same properties as above ([**6**], 5.5.7). One can then construct $\mathfrak{m}$ as before and check that it has the right properties. (In [**6**] one assumes G to be almost simple, but one generalises the results in question easily to our more general situation.)

For arbitrary (good) p deeper results on nilpotent elements are required. Let me first rephrase our goal in the light of the construction over $\mathbf{C}$. The crucial thing is to get the grading $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}(i)$ with the 'right' properties. We no longer want to define such a grading by taking the eigenspaces of an operator of the form $\mathrm{ad}(h)$ since then the eigenvalues are at best in $\mathbf{Z}/p\mathbf{Z}$ and not in $\mathbf{Z}$. Instead we want to use a homomorphism φ from the multiplicative group G_m to G (a 'one parameter group') and set for all $i \in \mathbf{Z}$

$$\mathfrak{g}(i) = \{\, x \in \mathfrak{g} \mid \mathrm{Ad}(\varphi(t))(x) = t^i x \text{ for all } t \in K,\ t \neq 0.\,\}. \tag{1}$$

This yields (for any φ) a grading $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}(i)$ of $\mathfrak{g}$ as restricted Lie algebra. (This means that $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ and $\mathfrak{g}(i)^{[p]} \subset \mathfrak{g}(pi)$ for all i and j. The G–invariance of $(\ ,\ )$ implies that $\mathfrak{g}(i) \perp \mathfrak{g}(j)$ for $i+j \neq 0$. It follows that $(\ ,\ )$ induces a perfect duality between $\mathfrak{g}(i)$ and $\mathfrak{g}(-i)$ and that $\dim(\mathfrak{g}(i)) = \dim(\mathfrak{g}(-i))$ for all i.

What we need is this:

Proposition. *There exists a one-parameter group φ such that the corresponding grading satisfies $e \in \mathfrak{g}(2)$ and $\mathfrak{c}_{\mathfrak{g}}(e) \subset \bigoplus_{i \geq 0} \mathfrak{g}(i)$ and $\dim C_G(e) = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1)$.*

If we have this then we can define $\mathfrak{m}$ by the same procedure as over $\mathbf{C}$. The inclusions $\mathfrak{g}(i)^{[p]} \subset \mathfrak{g}(pi)$ imply that

$$\mathfrak{m}^{[p]} \subset \bigoplus_{i \geq p} \mathfrak{g}(-i) \subset \mathfrak{m}.$$

It follows that $\mathfrak{m}$ is a unipotent restricted Lie subalgebra of $\mathfrak{g}$ and (for $p > 2$) that $(e, \mathfrak{m}^{[p]}) = 0$. The other required properties of $\mathfrak{m}$ follow as over $\mathbf{C}$.

If $p = 2$ then a modification is needed since it may happen that $(e, x^{[p]}) \neq 0$ for some $x \in \mathfrak{g}(-1)$. For $p = 2$ the semilinearity of the map $x \mapsto x^2 - x^{[2]}$ implies that $(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x, y]$ for all $x, y \in \mathfrak{g}$. It follows that $q(x) = (e, x^{[2]})$ defines a quadratic form on $\mathfrak{g}(-1)$ with associated bilinear form $\langle x, y\rangle = (e, [x, y])$. One now has to choose $\mathfrak{g}(-1)'$ as a maximal totally singular subspace of $\mathfrak{g}(-1)$ with respect to q. Then everything works as before.

7.9. The problem is to find φ as in Proposition 7.8. One can reduce to the case that either $G = \mathrm{GL}_n(K)$ for some $n \geq K$ or that G is almost simple not of type A. In that case Pommerening showed in [**39**] that the Bala-Carter parametrisation of nilpotent orbits (done over $\mathbf{C}$) works also over K. That parametrisation leads in a natural way to a homomorphism $\varphi : G_m \to G$ such that $e \in \mathfrak{g}(2)$ for the corresponding grading. The other required properties of the grading follow immediately from the results in the article [**49**] by Spaltenstein. The main point is a comparison with the orbit over $\mathbf{C}$ that has the same 'Bala-Carter data' as e and a proof that the orbits over K and over $\mathbf{C}$ have the same dimension. One has also to use that $\mathfrak{c}_{\mathfrak{g}}(e) = \mathrm{Lie}(C_G(e))$ in these cases, see [**6**], 1.14.

In [**41**] Premet gives another proof for the existence of φ as above (unaware of [**49**]).

7.10. Proposition 7.6 is proved in [**40**] for *faithful simple* modules if G satisfies (H1) and (H2). (For $p = 2$ also (H3) is required in [**40**]. That extra condition was removed in [**42**], 4.1.)

On the other hand, assuming (H3) one can avoid the restriction to faithful simple modules.

8. Rank Varieties and Premet's Theorem

8.1. Let $\mathfrak{g}$ be any restricted Lie algebra over K. Let $\chi \in \mathfrak{g}^*$. If $x \in \mathfrak{g}$, $x \neq 0$ satisfies $x^{[p]} = 0$ then Kx is a restricted Lie subalgebra of $\mathfrak{g}$ isomorphic to the restricted Lie algebra from Example 2b. We have

$$U_\chi(Kx) = U(Kx)/(x^p - \chi(x)^p) \simeq K[t]/((t - \chi(x))^p).$$

Let M be a $U_\chi(Kx)$–module. Since $(x - \chi(x))^p M = 0$, the Jordan normal form of x on M looks like

$$\text{matrix } (x_{|M}) = \begin{pmatrix} J_{a_1} & 0 & \dots & 0 \\ 0 & J_{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{a_r} \end{pmatrix}$$

where each

$$J_a = \begin{pmatrix} \chi(x) & 1 & 0 & \dots & 0 & 0 \\ 0 & \chi(x) & 1 & \dots & 0 & 0 \\ 0 & 0 & \chi(x) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \chi(x) & 1 \\ 0 & 0 & 0 & \dots & 0 & \chi(x) \end{pmatrix} \in M_a(K)$$

is a Jordan block of size a and where all a_i satisfy $1 \leq a_i \leq p$. Then M is a projective $U_\chi(Kx)$–module if and only if M is free over $U_\chi(Kx)$ if and only if $a_i = p$ for all i. Evidently, this occurs if and only if $\dim(\ker(x_{|M} - \chi(x))) = (\dim M)/p$.

Definition: If M is a $U_\chi(\mathfrak{g})$–module set

$$\Phi_{\mathfrak{g}}(M) = \{0\} \cup \{\, x \in \mathfrak{g} \mid x \neq 0,\ x^{[p]} = 0,\ \dim(\ker(x_{|M} - \chi(x))) > (\dim M)/p \,\}.$$

Then $\Phi_{\mathfrak{g}}(M)$ is called the *rank variety* of M.

8.2. Since the conditions in the definition of $\Phi_{\mathfrak{g}}(M)$ are closed we see $\Phi_{\mathfrak{g}}(M) \subset \mathfrak{g}$ is a (Zariski) closed set. It is clear from the definition that

$$\Phi_{\mathfrak{m}}(M) \;=\; \mathfrak{m} \cap \Phi_{\mathfrak{g}}(M) \tag{1}$$

for each restricted subalgebra $\mathfrak{m} \subset \mathfrak{g}$. If M' is a submodule of M, then one easily checks

$$\Phi_{\mathfrak{g}}(M) \;\subset\; \Phi_{\mathfrak{g}}(M') \cup \Phi_{\mathfrak{g}}(M/M'). \tag{2}$$

8.3. The importance of rank varieties comes (for us) from the following result proved by Friedlander and Parshall in [**19**], Thm. 6.4:

Theorem. *The module M is projective over $U_\chi(\mathfrak{g})$ if and only if $\Phi_{\mathfrak{g}}(M) = 0$.*

Here one direction is easy: Suppose that M is projective over $U_\chi(\mathfrak{g})$ and let $x \in \mathfrak{g}$ with $x \neq 0$ and $x^{[p]} = 0$. Since $U_\chi(\mathfrak{g})$ is free over $U_\chi(Kx)$ by Proposition 4.1, each projective $U_\chi(\mathfrak{g})$–module is projective over $U_\chi(Kx)$. Now use the comments before the theorem to get $x \notin \Phi_{\mathfrak{g}}(M)$.

The proof of the other direction is (much) more complicated. It was first proved for $\chi = 0$ in [**18**], Cor. 1.4 (that has to be combined with [**17**], Prop. 1.5). The proof relies on the main result in [**28**].

The main idea is to consider $\mathrm{Ext}^*_{U_\chi(\mathfrak{g})}(M, M)$ as a module over $\mathrm{Ext}^*_{U_0(\mathfrak{g})}(K, K)$. The direct sum of all $\mathrm{Ext}^{2n}_{U_0(\mathfrak{g})}(K, K)$ is a finitely generated commutative K–algebra. So its maximal spectrum is an algebraic variety, that we call the *cohomology variety* of $\mathfrak{g}$. The annihilator of the module $\mathrm{Ext}^*_{U_\chi(\mathfrak{g})}(M, M)$ is an ideal that defines a closed subset in this cohomology variety: the *support variety* of M. Now one shows on one hand that M is projective if and only if its support variety is 0 if and only if the support variety is finite. On the other hand there is a finite morphism from the support variety to $\Phi_{\mathfrak{g}}(M)$. The combination of these facts yields the theorem.

8.4. We now return to the situation from Sections 6 and 7: Let G be a connected reductive algebraic group over K and set $\mathfrak{g} = \mathrm{Lie}(G)$. Keep the other notations introduced in Section 6 and assume that (H1)–(H3) are satisfied (unless stated otherwise).

8.5. Premet's theorem (7.1) follows from the following result that he proves in [**42**], [**43**], [**44**] — in [**42**] and [**44**] for semisimple G just satisfying (H1), in [**43**], 4.3 for reductive G satisfying (H1) and (H2):

Theorem. *Let $\chi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$ with $x^{[p]} = 0$. If $\chi([x, \mathfrak{g}]) \neq 0$ then $x \notin \Phi_{\mathfrak{g}}(M)$ for all $U_\chi(\mathfrak{g})$–modules M.*

Note that the condition $\chi([x, \mathfrak{g}]) \neq 0$ is equivalent to $x \notin \mathfrak{c}_{\mathfrak{g}}(\chi)$. Any $\mathfrak{m}$ as in Theorem 7.1 satisfies $\mathfrak{m} \cap \mathfrak{c}_{\mathfrak{g}}(\chi) = 0$, hence $\mathfrak{m} \cap \Phi_{\mathfrak{g}}(M) = 0$ by Theorem 8.5. Now 8.2(1) yields $\Phi_{\mathfrak{m}}(M) = 0$. Then by Theorem 8.3 M is a projective $U_\chi(\mathfrak{m})$–module. This shows that Theorem 7.1 follows from Theorem 8.5. (On the other hand, Theorem 8.5 is the special case $\mathfrak{m} = Kx$ of Theorem 7.1.)

Remark. Before I begin discussing the proof of Theorem 8.5 let me mention a converse. Let $\chi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$ with $x^{[p]} = 0$. If $x \in \mathfrak{c}_{\mathfrak{g}}(\chi)$, then there exists a $U_\chi(\mathfrak{g})$–module M with $x \in \Phi_{\mathfrak{g}}(M)$. This is proved by Premet in [**43**] assuming (H1) and (H2).

8.6. Return to Theorem 8.5. Let $\chi \in \mathfrak{g}^*$ and M be a $U_\chi(\mathfrak{g})$–module, let $x \in \mathfrak{g}$ with $x^{[p]} = 0$ and $x \notin \mathfrak{c}_\mathfrak{g}(\chi)$. How can one prove that $x \notin \Phi_\mathfrak{g}(M)$? We first look at two special situations:

Case A : Suppose that $\chi(x) \neq 0$ and there exists $h \in \mathfrak{g}$ with $[h, x] = x$ and $h^{[p]} = h$. Then $\mathfrak{s} = Kh + Kx$ is a two dimensional restricted Lie subalgebra of $\mathfrak{g}$ isomorphic to the restricted Lie algebra from Example 1 studied in Sections 1 and 2. The calculations there show that each simple $U_\chi(\mathfrak{s})$–module E has dimension equal to p (recall that $\chi(x) \neq 0$) and that the kernel of $x - \chi(x)$ on E has dimension 1. This proves $x \notin \Phi_\mathfrak{s}(E)$, hence by 8.2(2) also $x \notin \Phi_\mathfrak{s}(N)$ for any $U_\chi(\mathfrak{s})$–module N. This yields $x \notin \Phi_\mathfrak{g}(M)$ by 8.2(1).

This case can actually be applied in most cases when $\chi(x) \neq 0$. If there is a homomorphism φ as in the discussion of the proof of Theorem 7.6 and if $p \neq 2$, then $h = (d\varphi\,(1))/2$ will work.

Case B : Suppose that there exists $y, z \in \mathfrak{g}$ with $y^{[p]} = 0 = z^{[p]}$, with $[x, y] = z$ and $[x, z] = [y, z] = 0$ such that $\chi(z) \neq 0$. Then $\mathfrak{s} = Kx + Ky + Kz$ is a restricted Lie subalgebra of $\mathfrak{g}$ isomorphic to the three dimensional Heisenberg Lie algebra from Example 3 studied in Section 4. The calculations there show that each simple $U_\chi(\mathfrak{s})$–module E has dimension equal to p (recall that $\chi(z) \neq 0$) and that the kernel of $x - \chi(x)$ on E has dimension 1. This yields $x \notin \Phi_\mathfrak{g}(M)$ arguing as in Case A.

8.7. These two special cases have the following in common: We have an element $y \in \mathfrak{g}$ with $\chi([x, y]) \neq 0$ such that the Lie algebra $\mathfrak{s}$ generated by x and y is “simple” and we use the explicitly known representation theory of $\mathfrak{s}$. The assumption $\chi([x, \mathfrak{g}]) \neq 0$ says of course that there always exists $y \in \mathfrak{g}$ with $\chi([x, y]) \neq 0$. In general there seems to be no hope that we can choose y such that $\mathfrak{s}$ is as “simple” as in these examples. However, in most cases one can find y such that the assumptions of the following lemma are satisfied.

Lemma. *Suppose that there exists $y \in \mathfrak{g}$ with $y^{[p]} = 0$ and $\mathrm{ad}(y)^2(\mathfrak{g}) \subset Ky$ such that $h = [y, x]$ satisfies*

$$\textit{either: } \quad [h, y] = 2y,\ h^{[p]} = h, \qquad\qquad \textit{or: } \quad [h, y] = 0,\ h^{[p]} = 0. \tag{1}$$

If $\chi(y) = 0$ and $\chi(h) \neq 0$ then $x \notin \Phi_\mathfrak{g}(M)$ for any $U_\chi(\mathfrak{g})$–module M.

We shall refer to the two possibilities in (1) as Case 1 and Case 2. Set

$$\mathfrak{a} = \{\, z \in \mathfrak{g} \mid [z, y] \in Ky \,\}.$$

That is a Lie subalgebra of $\mathfrak{g}$, the normaliser of Ky. The assumption $\mathrm{ad}(y)^2(\mathfrak{g}) \subset Ky$ says that $\mathrm{ad}(y)(\mathfrak{g}) \subset \mathfrak{a}$. We claim now that there exists $z \in \mathfrak{a}$ with

$$[h, x] = \begin{cases} -2x + z, & \text{in Case 1;} \\ z, & \text{in Case 2.} \end{cases}$$

Indeed, we set $z = [h, x] + cx$ (where $c = 2$ or $c = 0$ respectively) and then a simple calculation shows that $[y, z] = 0$, hence $z \in \mathfrak{a}$.

Note: If $z = 0$, then x, y, h span in Case 2 a Heisenberg Lie subalgebra as in Case B above and we get $x \notin \Phi_\mathfrak{g}(M)$ from that case. If $z = 0$ in Case 1, then x, y, h span a Lie subalgebra isomorphic to $\mathfrak{sl}_2(K)$ and the claim could be deduced from the results in Section 5. The point is now to show that crucial features of the representation theory of these three dimensional Lie algebras work even when $z \neq 0$.

Consider now M. The assumptions $y^{[p]} = 0$ and $\chi(y) = 0$ imply that y^p annihilates M. Set $M^0 = \{m \in M \mid y.m = 0\}$. A look at the Jordan normal form of y on M implies that $\dim(M) \leq p \cdot \dim(M^0)$. Choose a basis $m_1, m_2, \ldots, m_r$ of M^0.

Now the main point is to show that all $x^i.m_j$ with $0 \le i < p$ and $1 \le j \le r$ are linearly independent. If so, then the dimension estimate above shows that these $x^i.m_j$ are a basis for M. Then also the $(x - \chi(x))^i.m_j$ are a basis. It follows that the kernel of $x - \chi(x)$ on M has dimension equal to $r = \dim(M)/p$, hence that $x \notin \Phi_{\mathfrak{g}}(M)$.

The proof of the linear independence of the $x^i.m_j$ generalises the proof from Example 1 in Section 1 for the linear independence of the e_i. One wants to use induction on l to show that the $x^i.m_j$ with $i \le l$ are linearly independent and would like to have a formula similar to 1.5(2) describing the action of y on the $x^i.m_j$. Since we do not have precise information on z, we can compute $y.(x^i m_j)$ only modulo 'lower order' terms in some sense. But that turns out to be enough for our purposes.

We are going to skip the necessary calculations that the reader may reconstruct from the proof of Theorem 1.1 in [**42**], following Lemma 3.4.

8.8. So the question is now: How (and when) can we find y as in this lemma? Let us restrict from now on to the case where G is almost simple. (The reduction to the almost simple case is easy for G semisimple, see [**42**], shortly after Lemma 3.1. For arbitrary, reductive G some extra work is required, see [**43**], 4.3.)

So assume that G is almost simple. Now every root vector x_α with α *long* satisfies $\mathrm{ad}(x_\alpha)^2\mathfrak{g} \subset K x_\alpha$. (Use that $\mathrm{ad}(x_\alpha)^2\mathfrak{g}_\beta \subset \mathfrak{g}_{\beta+2\alpha}$ and apply the classification of root systems of rank 2.)

Let $\mathcal{O} = G.x_\alpha$ be the orbit under the adjoint action of G of any x_α with α long. This orbit is independent of the choice of α because all roots of the same length are conjugate under the Weyl group. The closure of $\overline{\mathcal{O}}$ of this orbit is just $\overline{\mathcal{O}} = \mathcal{O} \cup \{0\}$ because the stabiliser of the line $Kx_\alpha \subset \mathfrak{g}$ is a parabolic subgroup in G. If one orders nilpotent orbits in $\mathfrak{g}$ by the inclusion of their closures then $\mathcal{O}$ is minimal among the non-zero orbits. Therefore $\mathcal{O}$ is usually called the *minimal nilpotent orbit* of $\mathfrak{g}$.

Since G acts on $\mathfrak{g}$ by automorphisms also each $y \in \mathcal{O}$ satisfies $\mathrm{ad}(y)^2\mathfrak{g} \subset Ky$. Furthermore we have $y^{[p]} = 0$ since this is true for root vectors and hence for their conjugates. So our candidates for y as in Lemma 8.7 will be $y \in \mathcal{O}$. We need now:

Proposition. *If R is not of type C_n with $n \ge 1$ then there exists $y \in \mathcal{O}$ such that $\chi([x, y]) \ne 0$ and $\chi(y) = 0$.*

If we have that then we get Theorem 8.5 for G almost simple not of type C_n ($n \ge 1$). Take y as in this proposition and set $h = [y, x]$. Since $\mathrm{ad}(y)^2(\mathfrak{g}) \subset Ky$ there exists $a \in K$ with $[h, y] = 2ay$. (For $p = 2$ note that $\mathrm{ad}(y)^2 = 0$ since $y^{[2]} = 0$.) Replacing y by a scalar multiple we see that we can assume that $a \in \{0, 1\}$. It remains to be shown that $h^{[p]} = ah$. If so then all assumptions in Lemma 8.7 are satisfied and we get the Theorem. For the proof of $h^{[p]} = ah$ we refer now to [**42**], Lemma 3.4.

8.9. Proposition 8.8 is Proposition 3.3 in [**42**] and [**44**]. Its proof can be split into the following two steps:

Lemma. *Define $\chi' \in \mathfrak{g}^*$ by $\chi'(y) \equiv \chi([x, y])$. Then χ and χ' are linearly independent.*

Proof. Recall that $x \in \mathfrak{g}$ with $x^{[p]} = 0$ and $\chi([x, \mathfrak{g}]) \ne 0$. The second assumption implies that $\chi, \chi' \ne 0$. If they are linearly dependent, then there exists $a \in K$, $a \ne 0$ with $\chi' = a\,\chi$. Note that $\chi' = -x.\chi$ where we regard $\mathfrak{g}^*$ as a $\mathfrak{g}$–module dual to $\mathfrak{g}$ with the adjoint action. This is a restricted representation of $\mathfrak{g}$. Therefore $x^{[p]} = 0$ implies that x^p acts as 0, hence that $x^p.\chi = 0$. On the other hand $x.\chi = -\chi' = -a\chi$ yields $x^p.\chi = (-a)^p\chi \ne 0$, a contradiction.

8.10. Proposition. *Suppose that G is almost simple with R not of type C_n, $n \geq 1$. If $\chi_1, \chi_2 \in \mathfrak{g}^*$ are linearly independent, then there exists $y \in \mathcal{O}$ such that $\chi_1(y) = 0$ and $\chi_2(y) \neq 0$.*

This says that $\mathcal{O}$ has a certain 'separation property'. The original proof in [**42**] excluded only type $A_1 = C_1$. However, there is a sign error in the part of the proof of Lemma 3.2 in [**42**] that deals with the types C_n, $n \geq 2$. When I asked Kraft, whether there was not a geometric proof of Proposition 8.10, he and Wallach found one for the analogous result over **C**. They also discovered that the proposition could not hold for type C_n.

If R is of type C_n then Proposition 8.8 turns out to hold unless χ has the form $\chi(z) = (u, z)$ for some fixed $u \in \mathcal{O}$. In that case another argument can be used, see [**44**].

9. Centres

9.1. Over **C** the centre of the enveloping algebra of a semisimple Lie algebra is described using the Harish-Chandra homomorphism. We can define similarly in our situation a linear map $\pi : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ as the projection with kernel $\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+$; this requires just the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. However, this map cannot restrict to an injective homomorphism on the centre $Z(\mathfrak{g})$ (unless $\mathfrak{g} = \mathfrak{h}$), e.g., because $\pi(x_\alpha^p) = 0$ whilst $x_\alpha^p \in Z(\mathfrak{g})$. It turns out that the correct object to study is the algebra of G–invariants for the adjoint action

$$U(\mathfrak{g})^G = \{\, u \in U(\mathfrak{g}) \mid g.u = u \text{ for all } g \in G \,\}$$

instead of the centre. If we carry out the same construction over **C** then we get the whole centre. But here $U(\mathfrak{g})^G$ is a proper subalgebra (unless $\mathfrak{g} = \mathfrak{h}$) because, e.g., $x_\alpha^p \notin U(\mathfrak{g})^G$.

One shows as in characteristic 0 that π restricts to an algebra homomorphism on the 0 weight space of $U(\mathfrak{g})$ with respect to the adjoint action of T. Clearly $U(\mathfrak{g})^G$ is contained in this 0 weight space, so π restricts to an algebra homomorphism $U(\mathfrak{g})^G \longrightarrow U(\mathfrak{h})$. The first thing to observe now is:

Lemma. *The restriction of π to $U(\mathfrak{g})^G$ is injective.*

Proof. Let $u \in U(\mathfrak{g})^G$ with $\pi(u) = 0$. Then $u \in U(\mathfrak{g})\mathfrak{n}^+$ since u has weight 0. It follows that $uZ_\chi(\lambda) = 0$ for all $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$ and for all $\lambda \in \Lambda_\chi$, since $\mathfrak{n}^+$ and thus also u annihilate the standard generator v_λ of $Z_\chi(\lambda)$. Now Proposition 6.7 implies that u annihilates any simple $U_\chi(\mathfrak{g})$–module if $\chi(\mathfrak{n}^+) = 0$.

For general $\chi \in \mathfrak{g}^*$ there exists $g \in G$ with $(g\chi)(\mathfrak{n}^+) = 0$, by Lemma 6.6. Recall from 2.9 that the adjoint action of g induces an isomorphism $U_\chi(\mathfrak{g}) \xrightarrow{\sim} U_{g\chi}(\mathfrak{g})$. We get therefore all simple $U_\chi(\mathfrak{g})$–modules as follows: We take a simple $U_{g\chi}(\mathfrak{g})$–module E and change the module structure such that now each $x \in U(\mathfrak{g})$ acts as $g.x$ acted before. By the first paragraph of this proof u annihilates E under the old action. Now $g.u = u$ shows that u annihilates E also under the new action.

Therefore u annihilates all simple $\mathfrak{g}$–modules. Now apply (e.g.) Corollary 1 to Theorem 5.1 in [**7**] to conclude $u = 0$.

9.2. In order to describe the image of $U(\mathfrak{g})^G$ under π we need the 'dot action' of the Weyl group. We denote the Weyl group of G with respect to T by W. This group is generated by reflections s_α with $\alpha \in R$. The action of each s_α on $\mathfrak{h}^*$ is given by $s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$ with a fixed $h_\alpha \in \mathfrak{h}$. (One has $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(h_\alpha) = 2$; this determines h_α if $p \neq 2$.)

Our assumption (H1) implies that the h_α with α simple are linearly independent in $\mathfrak{h}$. We can therefore find $\rho \in \mathfrak{h}^*$ with $\rho(h_\alpha) = 1$ for all simple α. The dot action on $\mathfrak{h}^*$ of any $w \in W$ is now defined by

$$w \bullet \lambda = w(\lambda + \rho) - \rho.$$

We get in particular $s_\alpha \bullet \lambda = s_\alpha(\lambda) - \alpha$ for all simple α. Because already the s_α with α simple generate W, we see thus that the dot action of W is independent of the choice of ρ.

This dot action on $\mathfrak{h}^*$ yields also a dot action on $U(\mathfrak{h})$. Since $\mathfrak{h}$ is commutative we can identify $U(\mathfrak{h})$ with the symmetric algebra $S(\mathfrak{h})$, hence with the algebra of polynomial functions of $\mathfrak{h}^*$. Thinking of $f \in U(\mathfrak{h})$ as a function on $\mathfrak{h}^*$ we define $w \bullet f$ for $w \in W$ by $(w \bullet f)(\lambda) = f(w^{-1} \bullet \lambda)$. (For example, if α is a simple root and $h \in \mathfrak{h}$, then we get $s_\alpha \bullet h = s_\alpha(h) - \alpha(h)1$.)

9.3. We can now state the generalisation of Harish-Chandra's theorem to prime characteristic p, proved in [**33**] for almost simple G:

Theorem. *The restriction of* π *is an isomorphism* $U(\mathfrak{g})^G \xrightarrow{\sim} U(\mathfrak{h})^{W\bullet}$.

Remark. For almost simple G the only restriction in [**33**] is to exclude $G = \mathrm{SO}_{2n+1}(K)$ for $p = 2$. The arguments from [**33**] can be made to work for all reductive G satisfying (H1). A gap (mentioned in [**11**]) in the proof of Lemma 4.7 in [**33**] (where one should prove that $\operatorname{Spec} Z_1$ is normal) does not affect the proof of this theorem.

9.4. We have for each $\lambda \in \mathfrak{h}^*$ an algebra homomorphism $\mathrm{cen}_\lambda : U(\mathfrak{g})^G \to K$ that maps any $u \in U(\mathfrak{g})^G$ to $\pi(u)(\lambda)$ where we regard $\pi(u) \in U(\mathfrak{h})$ as a polynomial function on $\mathfrak{h}^*$. Our construction then shows that every $u \in U(\mathfrak{g})^G$ acts as multiplication by $\mathrm{cen}_\lambda(u)$ on each $Z_\chi(\lambda)$ with $\chi \in \mathfrak{g}^*$ such that $Z_\chi(\lambda)$ is defined (i.e., with $\chi(\mathfrak{n}^+) = 0$ and $\lambda \in \Lambda_\chi$). Theorem 9.3 implies easily (as in the corresponding situation over $\mathbf{C}$):

Corollary. *Let* $\lambda, \mu \in \mathfrak{h}^*$. *Then* $\mathrm{cen}_\lambda = \mathrm{cen}_\mu$ *if and only if* $\lambda \in W \bullet \mu$.

9.5. One direction in the proof of Theorem 9.3 and of Corollary 9.4 is not difficult: Let $\lambda \in \mathfrak{h}^*$ and choose $\chi \in \mathfrak{g}^*$ such that $Z_\chi(\lambda)$ is defined. Recall that v_λ is the standard generator of $Z_\chi(\lambda)$. One checks without too much effort for each simple root α and for each $u \in U(\mathfrak{g})^G$ that u acts on $x_{-\alpha}^{p-1} v_\lambda$ as multiplication by

$$s_\alpha(\pi(u))(\lambda - (p-1)\alpha) = \pi(u)(s_\alpha(\lambda + \alpha)) = \pi(u)(s_\alpha \bullet \lambda) = \mathrm{cen}_{s_\alpha \bullet \lambda}(u).$$

Since it acts as multiplication by $\mathrm{cen}_\lambda(u)$ on the whole module we get $\mathrm{cen}_\lambda = \mathrm{cen}_{s_\alpha \bullet \lambda}$ for all simple α. Because W is generated by these s_α, we get $\mathrm{cen}_\lambda = \mathrm{cen}_{w \bullet \lambda}$ for all $w \in W$. Since this holds for all $\lambda \in \mathfrak{h}^*$ we get $\pi(U(\mathfrak{g})^G) \subset U(\mathfrak{h})^{W\bullet}$.

9.6. It is more complicated to show that π maps $U(\mathfrak{g})^G$ onto $U(\mathfrak{h})^{W\bullet}$. Early work on this problem (by Humphreys and by Veldkamp) used 'reduction modulo p techniques' to attack this problem. That approach cannot work in all cases, but it turns out to work under our three hypotheses.

Let $G_\mathbf{Z}$ be a split reductive group scheme over $\mathbf{Z}$ with the same root data as G, let $T_\mathbf{Z}$ be a split maximal torus in $G_\mathbf{Z}$. For each $\mathbf{Z}$–algebra A write G_A and T_A for the group schemes over A that we get from $G_\mathbf{Z}$ and $T_\mathbf{Z}$ by extension of scalars from $\mathbf{Z}$ to A. Set then $\mathfrak{g}_A = \mathrm{Lie}(G_A)$ and $\mathfrak{h}_A = \mathrm{Lie}(T_A)$; these Lie algebras come with natural isomorphisms $\mathfrak{g}_\mathbf{Z} \otimes_\mathbf{Z} A \xrightarrow{\sim} \mathfrak{g}_A$ and $\mathfrak{h}_\mathbf{Z} \otimes_\mathbf{Z} A \xrightarrow{\sim} \mathfrak{h}_A$. (If G is semisimple, then our hypothesis (H1) implies that $\mathfrak{g}_\mathbf{Z}$ is the $\mathbf{Z}$–form of the corresponding semisimple complex Lie algebra described in [**24**], 25.2.)

We have a dot action of W on $(\mathfrak{h}_\mathbf{Z})^*$ given by $w \bullet \lambda = w(\lambda + \rho) - \rho$ where $\rho \in (\mathfrak{h}_\mathbf{Q})^*$ is half the sum of the positive roots. This leads as above to a dot action of W on $U(\mathfrak{h}_\mathbf{Z})$, hence by extension of scalars to one on each $U(\mathfrak{h}_A)$.

We can assume that $G_K = G$ and $T_K = T$, hence $\mathfrak{g}_K = \mathfrak{g}$ and $\mathfrak{h}_K = \mathfrak{h}$. The dot action of W on $U(\mathfrak{h})$ that we get by extension of scalars from that on $U(\mathfrak{h}_\mathbf{Z})$ coincides with the dot action introduced earlier.

Set $A = \mathbf{Z}_{(p)}$, the localisation of $\mathbf{Z}$ at p, and set $B = \mathbf{F}_p = A/pA$. The homomorphism π maps $U(\mathfrak{g}_B)^{G_B} \subset U(\mathfrak{g})^G$ injectively to $U(\mathfrak{h}_B)^{W\bullet} \subset U(\mathfrak{h})^{W\bullet}$. Since $K \supset B$ is flat we have $U(\mathfrak{g})^G = U(\mathfrak{g}_B)^{G_B} \otimes_B K$ and $U(\mathfrak{h})^{W\bullet} = U(\mathfrak{h}_B)^{W\bullet} \otimes_B K$, see [**29**], I.2.10(3). Therefore the surjectivity in Theorem 9.3 is equivalent to the surjectivity of $U(\mathfrak{g}_B)^{G_B} \to U(\mathfrak{h}_B)^{W\bullet}$.

We have a commutative diagram

$$\begin{array}{ccc} U(\mathfrak{g}_A)^{G_A} & \xrightarrow{\pi'} & U(\mathfrak{h}_A)^{W\bullet} \\ \varphi\downarrow & & \downarrow\psi \\ U(\mathfrak{g}_B)^{G_B} & \xrightarrow{\pi} & U(\mathfrak{h}_B)^{W\bullet} \end{array}$$

where φ and ψ arise from extension of scalars and where π' is defined analogously to π.

The kernel of φ is equal to $pU(\mathfrak{g}_A)^{G_A}$ since $U(\mathfrak{g}_A) \to U(\mathfrak{g}_B)$ has kernel $pU(\mathfrak{g}_A)$ and since $U(\mathfrak{g}_A)/U(\mathfrak{g}_A)^{G_A}$ is torsion free. By Harish-Chandra's theorem the analogue to π' over $\mathbf{Q}$ is an isomorphism. Therefore the cokernel of π' is a torsion module. These facts together with the injectivity of π imply: *If ψ is surjective, then π is surjective, hence π bijective and φ surjective.* This implies then Theorem 9.3 and that $U(\mathfrak{g})^G \simeq U(\mathfrak{g}_A)^{G_A} \otimes_A K$, hence (since $A \supset \mathbf{Z}$ is flat) that $U(\mathfrak{g})^G \simeq U(\mathfrak{g}_\mathbf{Z})^{G_\mathbf{Z}} \otimes_\mathbf{Z} K$.

There is an algebra automorphism of $U(\mathfrak{h}_A)$ that takes any $h \in \mathfrak{h}_A$ to $h - \rho(h)1$; similarly for $U(\mathfrak{h}_B)$. These automorphisms transform the dot action of W to the usual one and thus induce isomorphisms $U(\mathfrak{h}_A)^{W\bullet} \xrightarrow{\sim} U(\mathfrak{h}_A)^W$ and $U(\mathfrak{h}_B)^{W\bullet} \xrightarrow{\sim} U(\mathfrak{h}_B)^W$. Since these maps commute with base change we see that ψ is surjective if and only if the obvious map $U(\mathfrak{h}_A)^W \to U(\mathfrak{h}_B)^W$ is surjective. This will certainly follow when we can show that $U(\mathfrak{h}_B)^W = U(\mathfrak{h}_\mathbf{Z})^W \otimes_\mathbf{Z} B$.

Cor. 2 du th. 2 in [**12**] describes conditions for the last equality to hold. We have to apply his results to the lattice $M = \mathfrak{h}_\mathbf{Z}$ and to the root system $R^\vee \subset \mathfrak{h}_\mathbf{Z}$. One condition in [**12**] says: If there exists $\alpha \in R$ with $\alpha^\vee/2 \in M$ then $p \neq 2$. This condition follows in our situation from hypothesis (H1): The reduction modulo p of $\alpha^\vee$ is the h_α as in 9.2 and (H1) implies that $h_\alpha \neq 0$. If $\alpha^\vee/2 \in M$ then $h_\alpha \in 2\mathfrak{h}$, hence $p \neq 2$.

The second condition in [**12**] says that p should not be a "torsion prime". There are two kinds of torsion primes: There are the torsion primes of the root system $R^\vee$, which can be found in [**12**], Prop. 8. It turns out that all these primes are bad for R, hence excluded if we assume (H2). The second kind of torsion primes are those that divide the order of the cokernel of a certain map $i : M \to P(\mathcal{R})$ (in the notation from [**12**]). In our setting i can be described as follows: Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the simple roots. Then i can be identified with

$$\mathfrak{h}_\mathbf{Z} \to \mathbf{Z}^n, \qquad h \mapsto (\alpha_1(h), \alpha_2(h), \dots, \alpha_n(h)).$$

The cokernel of this map has no p torsion if and only if the corresponding map

$$\mathfrak{h} \to K^n, \qquad h \mapsto (\alpha_1(h), \alpha_2(h), \dots, \alpha_n(h))$$

is surjective, hence if and only if $\alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent when considered as elements in $\mathfrak{h}^*$. That however follows from (H3), see (e.g.) the argument later on in 11.2.

We see thus that under our hypotheses (H1)–(H3) not only Theorem 9.3 and Corollary 9.4 hold (also for reductive G), but also $U(\mathfrak{g})^G \simeq U(\mathfrak{g}_\mathbf{Z})^{G_\mathbf{Z}} \otimes_\mathbf{Z} K$.

9.7. In the quantum situation from 1.4 one can define a subalgebra analogous to $U(\mathfrak{g})^G$ and the prove a result similar to Theorem 9.3, see [**11**], Thm. 6.7.

According to Lemma 4.7 in [**33**] the whole centre $Z(\mathfrak{g})$ is generated by $U(\mathfrak{g})^G$ and $Z_0(\mathfrak{g})$ as an algebra. In [**11**], Theorem 6.4 a quantum analogue is proved. In a remark following that theorem the authors of [**11**] state that the proof in [**33**] contains a gap, but that their arguments work also in the prime characteristic case.

10. Standard Levi form

10.1. The study of simple $U_\chi(\mathfrak{g})$–modules reduces by Proposition 7.4 to the case where χ is nilpotent. By Lemma 6.6 we can assume, without loss of generality, that $\chi(\mathfrak{b}^+) = 0$. We know then by Proposition 6.7 that each simple $U_\chi(\mathfrak{g})$–module is the homomorphic image of some $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$. We have seen in 6.9 that some $Z_\chi(\lambda)$ have more than one maximal submodule. We want to look at an important case where this does not happen.
Definition: We say that χ has *standard Levi form* if and only if $\chi(\mathfrak{b}^+) = 0$ and there exists a subset I of the set of all simple roots such that

$$\chi(x_{-\alpha}) = \begin{cases} \neq 0, & \text{if } \alpha \in I, \\ 0, & \text{if } \alpha \in R \setminus I. \end{cases}$$

Remark. This definition goes back to [**20**], 3.1. If χ satisfies the definition, then we can choose the root vectors x_α so that χ is the inner product with $\sum_{\alpha \in I} x_\alpha$. So χ corresponds under our isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ to a regular nilpotent element in a Levi subalgebra in $\mathfrak{g}$. The classification of nilpotent orbits shows that in types A_n and B_2 every nilpotent χ is conjugate to one in standard Levi form. In all other types this is false.

10.2. Proposition. *If χ has standard Levi form then each $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$ has a unique maximal submodule.*

Proof. By considering weights we see $\chi([\mathfrak{n}^-, \mathfrak{n}^-]) = 0$ and $\chi(\mathfrak{n}^{-[p]}) = 0$. Thus χ defines a one dimensional $\mathfrak{n}^-$–module which is a $U_\chi(\mathfrak{n}^-)$–module. Since $\mathfrak{n}^-$ is unipotent there is a unique simple $U_\chi(\mathfrak{n}^-)$–module. The projective cover of this simple module is $U_\chi(\mathfrak{n}^-)$. Hence, as an $\mathfrak{n}^-$–module, $U_\chi(\mathfrak{n}^-)$ has a simple head. But as $\mathfrak{n}^-$–modules there is an isomorphism $Z_\chi(\lambda) \simeq U_\chi(\mathfrak{n}^-)$. Hence, as a $U_\chi(\mathfrak{g})$–module, $Z_\chi(\lambda)$ has a simple head.
Remark. The proof works equally well if we weaken our condition on χ and replace the assumption $\chi(\mathfrak{b}^+) = 0$ by $\chi(\mathfrak{n}^+) = 0$ since we never use that $\chi(\mathfrak{h}) = 0$. In particular, the proposition extends to the case where $\chi(\mathfrak{n}^-) = \chi(\mathfrak{n}^+) = 0$.

10.3. For χ in standard Levi form, let $L_\chi(\lambda)$ be the simple quotient of $Z_\chi(\lambda)$ for any $\lambda \in \Lambda_\chi$. We know now by Proposition 6.7 that each simple $U_\chi(\mathfrak{g})$–module is isomorphic to some $L_\chi(\lambda)$. The next question we should answer is when two such simple modules are isomorphic. Before we do this in general, we look at the two extreme cases, where $I = \emptyset$ or where I consists of all simple roots:

10.4. Let $I = \emptyset$. This means that $\chi = 0$. In this case $L_0(\lambda)/\mathfrak{n}^- L_0(\lambda)$ is one dimensional and $\mathfrak{h}$ acts on this space via λ. So we get $L_0(\lambda) \simeq L_0(\mu)$ if and only if $\lambda = \mu$. In other words, we have a bijection between Λ_0 and the set of isomorphism classes of simple $U_0(\mathfrak{g})$–modules.
Remark. Recall that $\chi = 0$ means that we are looking at the restricted representations of $\mathfrak{g}$. They were studied by Curtis in [**8**] which predates everything told so far on representations of $\mathfrak{g}$ for $\mathfrak{g}$ reductive. He found the classification of the simple modules just stated. Furthermore he proved that these simple modules can be extended to simple modules for G. (Here one needs assumption (H1), the simple connectedness of $\mathcal{D}G$; otherwise no restrictions on p are required here.)

A conjecture by Lusztig predicts (for p not too small) the characters (and thus dimensions) of the simple G–modules. By Curtis's theorem this would yield also the dimensions (and more) for the simple $U_0(\mathfrak{g})$–modules. The conjecture is known to be true for all p larger than some unknown bound depending on the root system R, see [**1**].

10.5. Assume I contains all simple roots. Then the example considered in 6.9 shows that $Z_\chi(s_\alpha \bullet \lambda) \simeq Z_\chi(\lambda)$ for all simple α. It follows that $Z_\chi(w \bullet \lambda) \simeq Z_\chi(\lambda)$ for all $w \in W$, hence $L_\chi(w \bullet \lambda) \simeq L_\chi(\lambda)$. On the other hand, if $L_\chi(\mu) \simeq L_\chi(\lambda)$, then $U(\mathfrak{g})^G$ acts on both modules via the same character. So Corollary 9.4 yields $\mu \in W \bullet \lambda$. So we have shown

$$L_\chi(\mu) \simeq L_\chi(\lambda) \iff \mu \in W \bullet \lambda.$$

Recall that χ is the inner product with a nilpotent element of the form $\sum_{\alpha \in I} x_\alpha$. This element is in our situation a regular nilpotent element in $\mathfrak{g}$. We call then χ a regular nilpotent element in $\mathfrak{g}^*$. The orbit of a regular nilpotent element in $\mathfrak{g}$ is dense in the nilpotent cone of $\mathfrak{g}$ and has therefore also dimension equal to $2 \dim \mathfrak{n}^-$. It follows that also $\dim(G.\chi) = 2 \dim \mathfrak{n}^-$. Therefore Proposition 7.6, the Kac-Weisfeiler conjecture, implies that $p^{\dim \mathfrak{n}^-}$ divides the dimension of each $U_\chi(\mathfrak{g})$–module. On the other hand, each baby Verma module $Z_\chi(\lambda)$ has dimension $p^{\dim \mathfrak{n}^-}$ and each simple $U_\chi(\mathfrak{g})$–module $L_\chi(\lambda)$ is the homomorphic image of $Z_\chi(\lambda)$. This shows:

Proposition ([**19**]). *Suppose $\chi \in \mathfrak{g}^*$ has standard Levi form and is regular nilpotent. Then each $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$ is simple. We have $Z_\chi(\lambda) \simeq Z_\chi(\mu)$ if and only if $\mu \in W \bullet \lambda$.*

Remarks. a) One can avoid here the use of Proposition 7.6 and go back directly to Corollary 7.2. One has to show that $\mathfrak{n}^- \cap \mathfrak{c}_\mathfrak{g}(\chi) = 0$. That follows (e.g.) from Springer's calculations in [**50**], Thm. 2.6.
b) For $\mathfrak{g} = \mathfrak{sl}_2$ the discussion of Case II in Section 5 for $p = 2$ shows that our proposition does not extend to that case, where (H1) and (H2) are satisfied, but (H3) is not. It is unknown (today, 7 Oct 1997) whether the proposition holds for G of type E_8 and $p = 5$, where (H1) and (H3) are satisfied, but not (H2).
c) This proposition is proved in [**19**], 4.2/3 for certain types and in [**20**], 2.2/3/4 in general (under slightly more restrictive assumptions on p). The irreducibility of the $Z_\chi(\lambda)$ in this situation is also proved for $G = \mathrm{SL}_n$ and $p > n$ in [**37**], Thm. 5, and claimed in general in [**32**], Lemma 1 and in [**38**], Thm. 2.

10.6. We return to the general case; consider an arbitrary subset I of the set of all simple roots. Set then $\mathfrak{g}_I$ equal to the direct sum of $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $\alpha \in R \cap \mathbf{Z}I$, set $\mathfrak{u}$ equal to the direct sum of all $\mathfrak{g}_\alpha$ with $\alpha > 0$, $\alpha \notin \mathbf{Z}I$, and $\mathfrak{u}'$ equal to the direct sum of all $\mathfrak{g}_\alpha$ with $\alpha < 0$, $\alpha \notin \mathbf{Z}I$. Both $\mathfrak{p} = \mathfrak{g}_I \oplus \mathfrak{u}$ and $\mathfrak{p}' = \mathfrak{g}_I \oplus \mathfrak{u}'$ are parabolic subalgebras of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_I$.

If $\chi \in \mathfrak{g}^*$ satisfies $\chi(\mathfrak{u}) = 0$, then we can extend any $U_\chi(\mathfrak{g}_I)$–module V to a $U_\chi(\mathfrak{p})$–module letting $\mathfrak{u}$ act by 0. We can then induce to get a $U_\chi(\mathfrak{g})$–module $\mathcal{Z}(V) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V$. Clearly $\mathcal{Z}$ is an exact functor. Similarly, if $\chi(\mathfrak{u}') = 0$, then we get an exact functor $\mathcal{Z}'$ by first extending V to $\mathfrak{p}'$, letting $\mathfrak{u}'$ act by 0, and then inducing: $\mathcal{Z}'(V) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p}')} V$.

We have also functors $M \mapsto M^\mathfrak{u}$ and $M \mapsto M^{\mathfrak{u}'}$ in the other direction, taking $U_\chi(\mathfrak{g})$–modules to $U_\chi(\mathfrak{g}_I)$–modules. Frobenius reciprocity yields easily that these functors are right adjoint to $\mathcal{Z}$ and $\mathcal{Z}'$ respectively (when defined): We have functorial isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}_I}(V, M^\mathfrak{u}) \xrightarrow{\sim} \mathrm{Hom}_\mathfrak{g}(\mathcal{Z}(V), M) \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{g}_I}(V, M^{\mathfrak{u}'}) \xrightarrow{\sim} \mathrm{Hom}_\mathfrak{g}(\mathcal{Z}'(V), M)$$

10.7. In the situation of Proposition 7.4 the functors $\mathcal{Z}$ and $M \mapsto M^\mathfrak{u}$ were inverse equivalences of categories. This is not true in general, but one can show:

PROPOSITION. *Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{u}) = \chi(\mathfrak{u}') = 0$. Then $E^{\mathfrak{u}}$ is a simple $U_\chi(\mathfrak{g}_I)$–module for each simple $U_\chi(\mathfrak{g})$–module E. The map $E \mapsto E^{\mathfrak{u}}$ induces a bijection between the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules and the isomorphism classes of simple $U_\chi(\mathfrak{g}_I)$–modules. The inverse map takes a simple $U_\chi(\mathfrak{g}_I)$–module V to the head of $\mathcal{Z}(V)$.*

This is proved (in a more general situation) in [**48**], Theorems 1.1 and 1.2 together with Corollary 1.4. If χ has standard Levi form and I is the set of simple roots with $\chi(x_{-\alpha}) \neq 0$, then the result was proved in [**20**], 3.2/4. (We shall look at the proof in that case in 11.7.) It is also contained (for more general $\mathfrak{g}$) in [**36**], Prop. 1.2.4. (There $\chi = 0$ is assumed, but the arguments there work equally well if one assumes just $\chi(\mathfrak{u}) = \chi(\mathfrak{u}') = 0$.) Finally, one can also check that one can apply the results from [**22**], Section 3.

10.8. Everything we have done for $\mathfrak{g}$ can also be done for $\mathfrak{g}_I$. For example, we can construct a baby Verma module for each $\chi \in \mathfrak{g}_I^*$ with $\chi(\mathfrak{g}_I \cap \mathfrak{n}^+) = 0$ and each $\lambda \in \Lambda_\chi$:

$$Z_{\chi,I}(\lambda) = U_\chi(\mathfrak{g}_I) \otimes_{U_\chi(\mathfrak{g}_I \cap \mathfrak{b}^+)} K_\lambda.$$

If $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$ then one checks easily that there is an isomorphism

$$Z_\chi(\lambda) \simeq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} Z_{\chi,I}(\lambda).$$

Assume now that $\chi \in \mathfrak{g}^*$ has standard Levi form and that I is the set of simple roots α with $\chi(x_{-\alpha}) \neq 0$. Then the restriction of χ to $\mathfrak{g}_I$ has still standard Levi form and is now regular nilpotent. So Proposition 10.5 implies that all $Z_{\chi,I}(\lambda)$ are irreducible and that $Z_{\chi,I}(\lambda) \simeq Z_{\chi,I}(\mu)$ if and only if $\mu \in W_I \bullet \lambda$ where $W_I = \langle s_\alpha \mid \alpha \in I \rangle$ is the Weyl group of $\mathfrak{g}_I$. Now Proposition 10.7 implies:

PROPOSITION. *Suppose that χ has standard Levi form and that $I = \{\alpha \in R \mid \chi(x_{-\alpha}) \neq 0\}$. Then $L_\chi(\lambda) \simeq L_\chi(\mu)$ if and only if $\mu \in W_I \bullet \lambda$.*

Remark. Note that one direction follows also from the example in 6.9 which shows that $Z_\chi(\lambda) \simeq Z_\chi(s_\alpha \bullet \lambda)$ for all $\alpha \in I$.

10.9. For all χ in standard Levi form and all $\lambda \in \Lambda_\chi$ let $Q_\chi(\lambda)$ denote the projective cover of $Z_\chi(\lambda)$ as a $U_\chi(\mathfrak{g})$–module. We write $[M : L]$ to denote the multiplicity of a simple module L as a composition factor of a module M.

LEMMA. *Suppose that $\chi \in \mathfrak{g}^*$ has standard Levi form and that $\lambda \in \Lambda_\chi$. Then*

$$\dim Q_\chi(\lambda) = p^{\dim \mathfrak{n}^+} \sum_{\mu \in \Lambda_\chi} [Z_\chi(\mu) : L_\chi(\lambda)].$$

Proof. The restriction of $Q_\chi(\lambda)$ to $U_\chi(\mathfrak{n}^+)$ is a projective module because the restriction of $U_\chi(\mathfrak{g})$ to $U_\chi(\mathfrak{n}^+)$ is free. Since $\chi(\mathfrak{n}^+) = 0$ and since $\mathfrak{n}^+$ is unipotent it follows from Corollary 3.4 that $Q_\chi(\lambda)$ is free over $U_\chi(\mathfrak{n}^+)$. Arguing as for 7.4(4) we get

$$\dim Q_\chi(\lambda) = p^{\dim \mathfrak{n}^+} \dim Q_\chi(\lambda)^{\mathfrak{n}^+}.$$

We have a natural isomorphism

$$\operatorname{Hom}_{\mathfrak{n}^+}(K, Q_\chi(\lambda)) \xrightarrow{\sim} Q_\chi(\lambda)^{\mathfrak{n}^+}.$$

On the other hand Frobenius reciprocity implies

$$\operatorname{Hom}_{\mathfrak{n}^+}(K, Q_\chi(\lambda)) \simeq \operatorname{Hom}_{\mathfrak{b}^+}(U_\chi(\mathfrak{b}^+) \otimes_{U_\chi(\mathfrak{n}^+)} K, Q_\chi(\lambda)).$$

Since $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}^+$ it acts trivially on the induced module $U_\chi(\mathfrak{b}^+)\otimes_{U_\chi(\mathfrak{n}^+)} K$. Considered as an $\mathfrak{h}$–module this induced module is isomorphic to $U_\chi(\mathfrak{h})$, hence to the direct sum of all K_μ with $\mu \in \Lambda_\chi$. It follows that

$$\mathrm{Hom}_{\mathfrak{b}^+}(U_\chi(\mathfrak{b}^+) \otimes_{U_\chi(\mathfrak{n}^+)} K, Q_\chi(\lambda)) \simeq \bigoplus_{\mu\in\Lambda_\chi} \mathrm{Hom}_{\mathfrak{b}^+}(K_\mu, Q_\chi(\lambda)).$$

Frobenius reciprocity yields for all μ

$$\mathrm{Hom}_{\mathfrak{b}^+}(K_\mu, Q_\chi(\lambda)) \simeq \mathrm{Hom}_{\mathfrak{g}}(Z_\chi(\mu), Q_\chi(\lambda)).$$

Finally, because $U_\chi(\mathfrak{g})$ is a symmetric algebra (see [**19**], Prop. 1.2), the projective cover $Q_\chi(\lambda)$ of $L_\chi(\lambda)$ is also the injective hull of $L_\chi(\lambda)$. This implies that

$$\dim \mathrm{Hom}_{\mathfrak{g}}(Z_\chi(\mu), Q_\chi(\lambda)) = [Z_\chi(\mu) : L_\chi(\lambda)].$$

Now the claim follows by combining the different equalities.
Remark. It is left to the reader to show that $Q_\chi(\lambda)$ considered as a $\mathfrak{b}^+$–module decomposes

$$Q_\chi(\lambda) \simeq_{\mathfrak{b}^+} \bigoplus_{\mu\in\Lambda_\chi} i(\mu)^{[Z_\chi(\mu):L_\chi(\lambda)]}$$

where $i(\mu)$ is the injective hull of K_μ as a $U_\chi(\mathfrak{b}^+)$–module. Furthermore one may show that $i(\mu)$ is isomorphic to $U_\chi(\mathfrak{n}^+)$ as a $\mathfrak{n}^+$–module while $\mathfrak{h}$ acts as the tensor product of the adjoint representation with a one dimensional representation such that $\mathfrak{h}$ acts via μ on the one dimensional subspace $U_\chi(\mathfrak{n}^+)^{\mathfrak{n}^+}$.

10.10. Proposition. *Suppose that $\chi \in \mathfrak{g}^*$ has standard Levi form and is regular nilpotent. Let $\lambda \in \Lambda_\chi$. Then $Q_\chi(\lambda)$ has length $|W \bullet \lambda|$. All composition factors of $Q_\chi(\lambda)$ are isomorphic to $L_\chi(\lambda)$.*

Proof. Each simple $U_\chi(\mathfrak{g})$–module has the form $Z_\chi(\mu)$. If it is a composition factor of $Q_\chi(\lambda)$, then $U(\mathfrak{g})^G$ has to act by the same character on $Z_\chi(\lambda)$ and on $Z_\chi(\mu)$, i.e., we have $\mathrm{cen}_\lambda = \mathrm{cen}_\mu$. Now Corollary 9.4 implies $\mu \in W \bullet \lambda$ and Proposition 10.5 implies that $Z_\chi(\mu)$ is isomorphic to $Z_\chi(\lambda)$. This yields the second claim of the proposition. The first one follows from Lemma 10.9 using Proposition 10.5 again.
Remark. This was first proved in [**19**], Thm. 4.3 for certain types and in [**20**], Thm. 2.4 in general.

10.11. Let $\chi \in \mathfrak{g}^*$ have standard Levi form and set $I = \{\alpha \in R \mid \chi(x_{-\alpha}) \neq 0\}$. Denote by $Q_{\chi,I}(\lambda)$ the projective cover of $Z_{\chi,I}(\lambda)$ as a $U_\chi(\mathfrak{g}_I)$–module. The preceding proposition (applied to $\mathfrak{g}_I$) says (for all $\lambda \in \Lambda_\chi$) that $Q_{\chi,I}(\lambda)$ has length $|W_I \bullet \lambda|$ with all composition factors isomorphic to $Z_{\chi,I}(\lambda)$. Set

$$Q_\chi^I(\lambda) = \mathcal{Z}(Q_{\chi,I}(\lambda)) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} Q_{\chi,I}(\lambda).$$

This is a $U_\chi(\mathfrak{g})$–module with a filtration of length $|W_I \bullet \lambda|$ with all quotients of subsequent terms in the filtration isomorphic to $Z_\chi(\lambda)$. One can now show:

Proposition. *Suppose that $\chi \in \mathfrak{g}^*$ has standard Levi form. Set $I = \{\alpha \in R \mid \chi(x_{-\alpha}) \neq 0\}$. Let $\lambda \in \Lambda_\chi$. Then $Q_\chi(\lambda)$ has a filtration where each quotient of subsequent terms in the filtration is isomorphic to some $Q_\chi^I(\mu)$. The number of factors isomorphic to a given $Q_\chi^I(\mu)$ is equal to $[Z_\chi(\mu) : L_\chi(\lambda)]$.*

This generalises results for $\chi = 0$ in [**23**] and [**27**]. It follows from Nakano's arguments in [**36**], §1.3. In order to get the claim on the multiplicities in the stated form, one has to show for each simple $U_\chi(\mathfrak{g}_I)$–module E that $\mathcal{Z}(E)$ and $\mathcal{Z}'(E^*)^*$ define the same class in the Grothendieck group of all $U_\chi(\mathfrak{g})$–modules. (Here $\mathcal{Z}'$ is actually not the $\mathcal{Z}'$ described in 10.6, but its analogue for χ replaced by $-\chi$.) This point will be discussed in the next section.

10.12. Consider again the case where χ is regular nilpotent. Proposition 10.10 says that distinct simple $U_\chi(\mathfrak{g})$–modules belong to distinct blocks of $U_\chi(\mathfrak{g})$. Therefore each $Q_\chi(\lambda)$ is a projective generator of the block belonging to $L_\chi(\lambda)$. It follows that this block is Morita equivalent to the algebra $\operatorname{End}_{\mathfrak{g}} Q_\chi(\lambda)$ (or rather to its opposite algebra where the order of multiplication is reversed). The dimension of $\operatorname{End}_{\mathfrak{g}} Q_\chi(\lambda)$ is equal to the multiplicity of $L_\chi(\lambda)$ as a composition factor of $Q_\chi(\lambda)$, hence equal to $|W \bullet \lambda|$ by Proposition 10.10.

PROPOSITION. *Suppose that $\chi \in \mathfrak{g}^*$ has standard Levi form and is regular nilpotent. Let $\lambda \in \Lambda_\chi$ with $\operatorname{Stab}_{W\bullet} \lambda = 1$. Then*

$$\operatorname{End}_{\mathfrak{g}} Q_\chi(\lambda) \simeq C$$

where C is the coinvariant algebra of W, that is $S(\mathfrak{h})/(S(\mathfrak{h})^W_+)$.

Remarks. a) This is a special case of an unpublished result by Soergel and me computing the endomorphism algebras of projective indecomposables in the 'top restricted alcove'. We proved the more general result first for $\chi = 0$ where it is contained (with a different proof) in [**1**], Prop. 19.8. We later realized that the proof works more generally. It imitates Bernstein's proof of Soergel's determination of the endomorphisms of the antidominant projectives (as described in Soergel's lectures at this meeting). We assume that p is greater than the Coxeter number, but that bound can probably be improved. (This does not influence the proposition as stated, since the existence of λ with trivial stabiliser implies that p has to satisfy that bound.)
b) Premet has announced a more general result dealing with all λ.

11. Graded Structures

11.1. Let $X = X(T)$ denote the character group of T. This is a free Abelian group of rank equal to $\dim T$. It contains the subgroup $\mathbf{Z}R$ generated by the roots. Also this subgroup is a free Abelian group; its rank is equal to the rank of R.

Each $\lambda \in X$ is a homomorphism of algebraic groups from T to the multiplicative group. Therefore its differential $d\lambda : \mathfrak{h} \to K$ is a homomorphism of restricted Lie algebras and satisfies $d\lambda(h^{[p]}) = d\lambda(h)^p$ for all $h \in \mathfrak{h}$. This means that $d\lambda \in \Lambda_0$ in the notation from Section 6. The map $\lambda \mapsto d\lambda$ has kernel pX and induces a bijection

$$X/pX \xrightarrow{\sim} \Lambda_0.$$

(This holds for arbitrary tori; it suffices to prove it in the case of the multiplicative group.) If $\chi \in \mathfrak{h}^*$ is arbitrary and $\mu \in \Lambda_\chi$, then $\Lambda_\chi = \mu + \Lambda_0 = \{\mu + d\lambda \mid \lambda \in X\}$.

11.2. Recall that we always assume that $\mathfrak{g}$ satisfies (H1)–(H3). This was not needed above, but now we have to use (H1) and (H3) to show that

$$\mathbf{Z}R \cap pX = p\mathbf{Z}R. \tag{1}$$

Well, hypothesis (H1) implies that the $h_\alpha = [x_\alpha, x_{-\alpha}]$ with α simple are linearly independent. We get from (H3) an isomorphism $\varphi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ of G–modules. A simple calculation shows that $\varphi(h_\alpha) = \varphi(x_\alpha)(x_{-\alpha}) \cdot d\alpha$. Since $\varphi(x_\alpha) \neq 0$ and since $\varphi(x_\alpha)$ vanishes on all $\mathfrak{g}_\beta$ with

$\beta \neq -\alpha$, we have $\varphi(x_\alpha)(x_{-\alpha}) \neq 0$. This implies that also the $d\alpha$ with α simple are linearly independent. The claim follows because these α are a basis of $\mathbf{Z}R$.

One shows now easily for any subset I of the set of simple roots that

$$\mathbf{Z}I \cap pX \;=\; p\mathbf{Z}I. \tag{2}$$

11.3. The algebra $U(\mathfrak{g})$ is $\mathbf{Z}R$–graded where

$$\deg(x_\alpha) = \alpha \quad \text{and} \quad \deg(h_i) = 0$$

Recall that (for all $\chi \in \mathfrak{g}^*$)

$$\begin{aligned} U_\chi(\mathfrak{g}) &= U(\mathfrak{g}) \,/\, (x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}) \\ &= U(\mathfrak{g}) \,/\, (h^p - h^{[p]} - \chi(h)^p,\, x_\alpha^p - \chi(x_\alpha)^p \mid h \in \mathfrak{h},\, \alpha \in R). \end{aligned}$$

The elements $h^p - h^{[p]} - \chi(h)^p$ have degree zero whilst x_α^p has degree $p\alpha$ and $\chi(x_\alpha)^p$ has degree zero. This shows that we obtain a natural $\mathbf{Z}R$–grading on $U_\chi(\mathfrak{g})$ if $\chi(x_\alpha) = 0$ for all $\alpha \in R$.

In general, we can give $U_\chi(\mathfrak{g})$ a natural grading by $\mathbf{Z}R/\mathbf{Z}R'$ where $R' = \{\alpha \in R \mid \chi(x_\alpha) \neq 0\}$. In the situation of χ having standard Levi form this turns $U_\chi(\mathfrak{g})$ into a $\mathbf{Z}R/\mathbf{Z}I$–graded algebra where I is the set of simple roots α with $\chi(x_{-\alpha}) \neq 0$. Note that we get in this case a grading by a free Abelian group of finite rank because $\mathbf{Z}I$ is generated by a subset of a basis of $\mathbf{Z}R$.

11.4. Fix from now on $\chi \in \mathfrak{g}^*$ having standard Levi form and let I be as above. (The first definitions to come could still be carried out in a more general setting, with a few modifications if $\chi(\mathfrak{h}) \neq 0$.)

We are going to study $U_\chi(\mathfrak{g})$–modules that are graded by the Abelian group $X/\mathbf{Z}I \supset \mathbf{Z}R/\mathbf{Z}I$. So we are looking at a (as always: finite dimensional) $U_\chi(\mathfrak{g})$–module M with a direct sum decomposition $M = \bigoplus_{\nu \in X/\mathbf{Z}I} M^\nu$ such that $\mathfrak{h}.M^\nu \subset M^\nu$ and $x_\alpha.M^\nu \subset M^{\nu+\alpha}$ for all $\alpha \in R$. (If we wanted to be very precise we should have written here $M^{\nu+(\alpha+\mathbf{Z}I)}$. Usually we do not.)

If M is such a graded module, then we denote (for each $\mu \in X/\mathbf{Z}I$) by $M\langle\mu\rangle$ the same module with the grading shifted by μ, i.e, with $(M\langle\mu\rangle)^\nu = M^{\nu-\mu}$ for all ν. If $\lambda \in X$, then we usually write $M\langle\lambda\rangle$ instead of $M\langle\lambda + \mathbf{Z}I\rangle$.

Let $\mathcal{F}$ denote the forgetful functor that takes each graded $U_\chi(\mathfrak{g})$–module to the underlying $U_\chi(\mathfrak{g})$–module. We have clearly $\mathcal{F}(M\langle\mu\rangle) = \mathcal{F}(M)$ for all M and μ.

11.5. Each graded piece M^ν of such an $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module M is a $U_\chi(\mathfrak{h})$–module. It has therefore a decomposition $M^\nu = \bigoplus_{\lambda \in \Lambda_0} M_\lambda^\nu$ into weight spaces. (Note that $\Lambda_0 = \Lambda_\chi$ since $\chi(\mathfrak{h}) = 0$.) We have then $x_\alpha.M_\lambda^\nu \subset M_{\lambda+d\alpha}^{\nu+\alpha}$ for all roots α. It is therefore clear that (for each $\mu \in \Lambda_0$)

$$M_{[\mu]} = \sum_{\nu \in X} M_{\mu+d\nu}^{\nu+\mathbf{Z}I}$$

is a graded submodule of M. Furthermore, one checks easily that M is the direct sum of all $M_{[\mu]}$ with μ running over a suitable system of representatives. A simple calculation shows for all $\lambda \in X$ and all $\mu \in \Lambda_0$ that

$$(M\langle\lambda\rangle)_{[\mu]} \;=\; (M_{[\mu+d\lambda]})\langle\lambda\rangle. \tag{1}$$

Let $\mathcal{C}$ denote the category of all $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules M with $M = M_{[0]}$.

A look at (1) shows (for any $\lambda \in X$) that the functor $M \mapsto M\langle\lambda\rangle$ is an equivalence of categories from $\mathcal{C}$ to the category of all $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules N with $N = N_{[-d\lambda]}$.

The category of all $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules is the direct sum of categories of this form. Therefore we do not lose anything by restricting ourselves to $\mathcal{C}$.
Remark. A category like $\mathcal{C}$ was first introduced in the case $\chi = 0$ in [**27**] under the name of $\mathbf{u}_1$–T–modules. Here $\mathbf{u}_1$ is just the notation used in [**27**] for the restricted enveloping algebra. Instead of X–gradings the definition in [**27**] involves a T–module structure; but that amounts to the same since an action of T leads to an X–grading by taking the weight spaces for T as the graded pieces (and vice versa). The 'compatibility' condition $M = M_{[0]}$ corresponds to the condition there that $\mathfrak{h} = \mathrm{Lie}(T)$ has to act by the derived action of the T–action.

For arbitrary χ (in standard Levi form) the corresponding definition appears in [**20**], Section 3 as (T', A_χ)–modules. Here A_χ is the notation used in [**20**] for $U_\chi(\mathfrak{g})$ and T' is the intersection of the kernels of the $\alpha \in I$ in T. This is a diagonalisable algebraic group with character group $X/\mathbf{Z}I$. [Here one uses 11.2(2).] So a T'–action is the same as a grading by $X/\mathbf{Z}I$. The condition $M = M_{[0]}$ can be expressed in term of the action of $\mathrm{Lie}(T')$.

Forget for a second that we assume χ to have standard Levi form. Suppose instead $\chi(\mathfrak{n}^+) = \chi(\mathfrak{n}^-) = 0$. Then $U_\chi(\mathfrak{g})$ is X–graded and we can consider X–graded $U_\chi(\mathfrak{g})$–modules $M = \bigoplus_{\nu \in X} M^\nu$. Pick $\lambda \in \Lambda_\chi$. Then the condition $M = M_{[0]}$ above has to be replaced by the condition $M^\nu = M^\nu_{\lambda + d\nu}$. One gets then as $\mathcal{C}$ the category $\mathcal{C}_A$ from [**1**], 2.3, for $A = k = K$ and $\pi : U^0 \to A$ (as in [**1**]) equal to the homomorphism $U_\chi(\mathfrak{h}) \to K$ defined by λ.

11.6. We define for each $\lambda \in X$ an $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module $\widehat{Z}_\chi(\lambda)$ with

$$\mathcal{F}(\widehat{Z}_\chi(\lambda)) \simeq Z_\chi(d\lambda) \tag{1}$$

as follows: We take the basis of $Z_\chi(d\lambda)$ as in 6.8 and put any $\prod_{\alpha>0} x_{-\alpha}^{a(\alpha)} v_\lambda$ into degree $\lambda - \sum_{\alpha>0} a(\alpha)\alpha + \mathbf{Z}I$. One checks that this is a grading as a $U_\chi(\mathfrak{g})$–module. It is then clear that $\widehat{Z}_\chi(\lambda)$ belongs to $\mathcal{C}$. (More systematically, one should introduce $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{b}^+)$–modules and define then an induction functor in the graded setting, see the analogous construction in [**1**], 2.6. Then $\widehat{Z}_\chi(\lambda)$ is induced from K_λ which is the $U_\chi(\mathfrak{b}^+)$–module $K_{d\lambda}$ put into degree $\lambda + \mathbf{Z}I$.)

One checks easily that one has for all $\lambda, \mu \in X$ an isomorphism

$$\widehat{Z}_\chi(\lambda + p\mu) \simeq \widehat{Z}_\chi(\lambda)\langle p\mu\rangle. \tag{2}$$

We have on $X/\mathbf{Z}I$ an order relation $\le$ such that $\mu \le \nu$ if and only if there exist integers $m_\alpha \ge 0$ with $\nu - \mu = \sum_\alpha m_\alpha \alpha + \mathbf{Z}I$ where α runs over the simple roots not in I. (The cosets modulo $\mathbf{Z}I$ of these α are linearly independent in $X/\mathbf{Z}I$; this shows that $\le$ is indeed an order relation.)

11.7. Recall the notations $\mathfrak{g}_I$, $\mathfrak{u}$, $\mathfrak{u}'$, etc. introduced before Proposition 10.7. If M is an $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module then each M^ν is a $\mathfrak{g}_I$–submodule of M. A look at the construction of the grading on $\widehat{Z}_\chi(\lambda)$ shows for all λ that $\widehat{Z}_\chi(\lambda)^\mu \neq 0$ implies $\mu \le \lambda + \mathbf{Z}I$ and that

$$\widehat{Z}_\chi(\lambda)^{\lambda + \mathbf{Z}I} \simeq_{\mathfrak{g}_I} Z_{\chi,I}(d\lambda). \tag{1}$$

The simplicity of this $\mathfrak{g}_I$–module implies that each proper graded submodule of $\widehat{Z}_\chi(\lambda)$ is contained in the direct sum of the $\widehat{Z}_\chi(\lambda)^\mu$ with $\mu \neq \lambda + \mathbf{Z}I$. Therefore $\widehat{Z}_\chi(\lambda)$ has a unique maximal graded submodule. Let $\widehat{L}_\chi(\lambda)$ denote the factor module (in $\mathcal{C}$) of $\widehat{Z}_\chi(\lambda)$ by that maximal graded submodule. Then $\widehat{L}_\chi(\lambda)$ is a simple object in $\mathcal{C}$. It is not difficult to see that each simple object in $\mathcal{C}$ is isomorphic to some $\widehat{L}_\chi(\lambda)$. The construction shows that the canonical surjection $\widehat{Z}_\chi(\lambda) \to \widehat{L}_\chi(\lambda)$ is an isomorphism on the homogeneous part of degree $\lambda + \mathbf{Z}I$. We get therefore

$$\widehat{L}_\chi(\lambda)^{\lambda+\mathbf{Z}I} \simeq_{\mathfrak{g}_I} Z_{\chi,I}(d\lambda).$$

Formula 11.6(2) implies (for all $\lambda, \mu \in X$)

$$\widehat{L}_\chi(\lambda + p\mu) \simeq \widehat{L}_\chi(\lambda)\langle p\mu\rangle. \tag{2}$$

LEMMA. *We have $\mathcal{F}(\widehat{L}_\chi(\lambda)) \simeq L_\chi(d\lambda)$ for all $\lambda \in X$.*

Proof. It is clear that $\mathcal{F}(\widehat{L}_\chi(\lambda))$ is homomorphic image of $Z_\chi(d\lambda)$. Therefore it suffices to show that $\mathcal{F}(\widehat{L}_\chi(\lambda))$ is a simple $\mathfrak{g}$–module. Well, any non-zero $\mathfrak{g}$–submodule M of $\widehat{L}_\chi(\lambda)$ satisfies $M^{\mathfrak{n}^+} \neq 0$. It therefore suffices to show that $\widehat{L}_\chi(\lambda)^{\mathfrak{n}^+} \subset \widehat{L}_\chi(\lambda)^{\lambda+\mathbf{Z}I}$. However, since $\mathfrak{n}^+$ is graded, so is $\widehat{L}_\chi(\lambda)^{\mathfrak{n}^+}$, and any $v \in \widehat{L}_\chi(\lambda)^{\mathfrak{n}^+}$ of weight $\mu < \lambda + \mathbf{Z}I$ generates a proper graded submodule of $\widehat{L}_\chi(\lambda)$, hence is 0.

Remark. The same type of argument shows that $\widehat{L}_\chi(\lambda)^{\mathfrak{u}} = \widehat{L}_\chi(\lambda)^{\lambda+\mathbf{Z}I}$; we get thus a proof of Proposition 10.7 in the present situation, basically the same proof as in [**20**].

11.8. We have just seen that each simple $U_\chi(\mathfrak{g})$–module E is 'gradable', that is, there is some M in $\mathcal{C}$ with $\mathcal{F}(M) \simeq E$. This is the special case of a more general result on modules over graded Artin algebras.

In [**21**] Gordon and Green study $\mathbf{Z}$–graded modules over $\mathbf{Z}$–graded Artin algebras. Their arguments extend to gradings by any free Abelian group of finite rank. So their results can be applied to $\mathbf{Z}R/\mathbf{Z}I$–graded modules over the $\mathbf{Z}R/\mathbf{Z}I$–graded algebra $U_\chi(\mathfrak{g})$. It is then not difficult to extend them to $X/\mathbf{Z}I$–graded modules over $U_\chi(\mathfrak{g})$: If M is such a module, then each

$$M^{[\mu]} = \sum_{\lambda\in\mathbf{Z}R} M^{\mu+\lambda+\mathbf{Z}I}$$

with $\mu \in X$ is a graded submodule, and M is the direct sum of all $M^{[\mu]}$ with μ running over representatives for $X/\mathbf{Z}R$. For each μ the category of all M with $M = M^{[\mu]}$ is isomorphic (via $M \mapsto M\langle\mu\rangle$) to the category of all $\mathbf{Z}R/\mathbf{Z}I$–graded modules over $U_\chi(\mathfrak{g})$.

The general results from [**21**] imply that not only simple $U_\chi(\mathfrak{g})$–modules but also projective indecomposable modules are gradable. A module M in $\mathcal{C}$ is simple (semisimple, projective, indecomposable) in $\mathcal{C}$ if and only if $\mathcal{F}(M)$ is simple (semisimple, projective, indecomposable). If M and M' are indecomposable modules in $\mathcal{C}$ with $\mathcal{F}(M) \simeq \mathcal{F}(M')$, then there exists $\lambda \in X$ with $M' \simeq M\langle\lambda\rangle$.

11.9. We next want to describe when $\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu)$. This requires the introduction of affine Weyl groups. The usual Weyl group W acts on X. The action of a reflection s_α with $\alpha \in R$ has the form $s_\alpha(\mu) = \mu - \langle\mu, \alpha^\vee\rangle\alpha$ where $\alpha^\vee$ is the coroot of α. We introduce for all $r \in \mathbf{Z}$ the affine reflection $s_{\alpha,rp}$ by $s_{\alpha,rp}(\mu) = \mu - (\langle\mu, \alpha^\vee\rangle - rp)\alpha$. This is a reflection with respect to the hyperplane $\langle\lambda, \alpha^\vee\rangle = rp$. Define now the *affine Weyl group* W_p as the group generated by all $s_{\alpha,rp}$ with $\alpha \in R$ and $r \in \mathbf{Z}$. One can also describe W_p as the group generated by W and by all translations by $p\beta$ with $\beta \in R$.

Let $W_{I,p}$ denote the subgroup of W_p generated by W_I and by all translations by $p\alpha$ with $\alpha \in I$. Equivalently, this is the subgroup generated by all $s_{\beta,rp}$ with $\beta \in R \cap \mathbf{Z}I$ and $r \in \mathbf{Z}$.

We use the dot action of W_p on X given by $w \bullet \lambda = w(\lambda + \rho) - \rho$ where ρ is now half the sum of the positive roots (taken possibly in $X \otimes_{\mathbf{Z}} \mathbf{Q}$). If w is a translation then clearly $w \bullet \lambda = w\lambda$. If α is a simple root, then $s_\alpha \bullet \lambda = s_\alpha(\lambda) - \alpha$. (This shows that $W_p \bullet X = X$ even if $\rho \notin X$. These formulas imply also that the dot action is compatible with the earlier one on $\mathfrak{h}^*$: We have $w \bullet (d\lambda) = d(w \bullet \lambda)$ for all $w \in W$ and $\lambda \in X$.)

Proposition. *Let $\lambda, \mu \in X$. Then*

$$\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu) \qquad \Longleftrightarrow \qquad \widehat{Z}_\chi(\lambda) \simeq \widehat{Z}_\chi(\mu) \qquad \Longleftrightarrow \qquad \mu \in W_{I,p} \bullet \lambda.$$

Proof. If $\mu = \lambda + p\alpha$ with $\alpha \in I$, then $\widehat{Z}_\chi(\mu) \simeq \widehat{Z}_\chi(\lambda)\langle p\alpha\rangle \simeq \widehat{Z}_\chi(\lambda)$ since $p\alpha + \mathbf{Z}I = 0 + \mathbf{Z}I$. If $\alpha \in I$ and $\langle \lambda, \alpha^\vee \rangle = mp + a$ with $a, m \in \mathbf{Z}$ and $0 \le a < p$, then the construction in 6.9 actually yields an isomorphism $\widehat{Z}_\chi(\lambda - (a+1)\alpha) \xrightarrow{\sim} \widehat{Z}_\chi(\lambda)$ in $\mathcal{C}$. We have $\lambda - (a+1)\alpha = s_\alpha \bullet \lambda + mp\alpha$, hence $\widehat{Z}_\chi(s_\alpha \bullet \lambda) \simeq \widehat{Z}_\chi(s_\alpha \bullet \lambda + mp\alpha) \simeq \widehat{Z}_\chi(\lambda)$. This implies that $\widehat{Z}_\chi(\lambda) \simeq \widehat{Z}_\chi(\mu)$ whenever $\mu \in W_{I,p} \bullet \lambda$.

That $\widehat{Z}_\chi(\lambda) \simeq \widehat{Z}_\chi(\mu)$ implies $\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu)$ is obvious. So it remains to be shown that $\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu)$ implies $\mu \in W_{I,p} \bullet \lambda$.

Note first that $\widehat{L}_\chi(\lambda)$ determines $\lambda + \mathbf{Z}I$ as the largest $\nu \in X/\mathbf{Z}I$ with $\widehat{L}_\chi(\lambda)^\nu \ne 0$. Furthermore $\widehat{L}_\chi(\lambda)$ determines the $\mathfrak{g}_I$–module $Z_{\chi,I}(d\lambda)$ as the graded piece $\widehat{L}_\chi(\lambda)^{\lambda+\mathbf{Z}I}$. Therefore $\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu)$ implies $\mu - \lambda \in \mathbf{Z}I$ and $Z_{\chi,I}(d\lambda) \simeq Z_{\chi,I}(d\mu)$. The second condition yields $d\mu \in W_I \bullet (d\lambda)$ by Proposition 10.5 applied to $\mathfrak{g}_I$, hence $\mu \in W_I \bullet \lambda + pX$. Pick $w \in W_I$ with $\mu - w \bullet \lambda \in pX$. Then $\lambda - w \bullet \lambda \in \mathbf{Z}I$ since $w \in W_I$. We know already that $\mu - \lambda \in \mathbf{Z}I$ and get therefore $\mu - w \bullet \lambda \in \mathbf{Z}I$. But this difference is also in pX. So 11.2(2) yields $\mu - w \bullet \lambda \in p\mathbf{Z}I$, hence $\mu \in W_{I,p} \bullet \lambda$.

11.10. Proposition 11.9 says that the simple modules in $\mathcal{C}$ are parametrised by the orbits of $W_{I,p}$ on X. The general theory of reflection groups says that a fundamental domain for the action of $W_{I,p}$ on $X_\mathbf{R} = X \otimes_\mathbf{Z} \mathbf{R}$ is given by

$$C_I = \{\, \lambda \in X_\mathbf{R} \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p, \quad \forall \alpha \in R^+ \cap \mathbf{Z}I \,\}.$$

So the simple modules in $\mathcal{C}$ can be parametrised by $C_I \cap X$.

11.11. The next result we need is the *linkage principle:*

Proposition. *If $\widehat{L}_\chi(\mu)$ is a composition factor of $\widehat{Z}_\chi(\lambda)$ then $\mu \in W_p \bullet \lambda$.*

The proof to be sketched in the next subsections follows the approach in [**15**] and in [**1**], 5.6–10. It actually yields a *strong linkage principle:* If we assume in the proposition that $\lambda, \mu \in C_I \cap X$, then we get $\mu \uparrow \lambda$ in the notations from [**29**], II.6.4.

It is clear by looking at the grading that $\widehat{L}_\chi(\lambda)$ is a composition factor of $\widehat{Z}_\chi(\lambda)$ with multiplicity 1 and that all other composition factors $\widehat{L}_\chi(\mu)$ satisfy $\mu + \mathbf{Z}I < \lambda + \mathbf{Z}I$. We want to use induction over $(\lambda - \mu) + \mathbf{Z}I$ to prove the proposition. This requires some preparations.

11.12. For any $w \in W$ let $w\mathfrak{n}^+$ be the direct sum of all $\mathfrak{g}_{w\alpha}$ with $\alpha > 0$; set $w\mathfrak{b}^+ = \mathfrak{h} \oplus w\mathfrak{n}^+$. Then $w\mathfrak{n}^+$ is the image of $\mathfrak{n}^+$ under the adjoint action of a representative of $w \in W = N_G(T)/T$ in $N_G(T)$; similarly for $w\mathfrak{b}^+$. We have $\chi(w\mathfrak{b}^+) = 0$ if and only if $w \in W^I$ where

$$W^I = \{\, w \in W \mid w^{-1}(\alpha) > 0 \text{ for all } \alpha \in I \,\}. \tag{1}$$

If $w \in W^I$ then each $\lambda \in X$ defines a one dimensional $U_\chi(w\mathfrak{b}^+)$–module $K_{d\lambda}$ and then an induced $U_\chi(\mathfrak{g})$–module

$$Z^w_\chi(d\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(w\mathfrak{b}^+)} K_{d\lambda}. \tag{2}$$

There is a unique structure as an $X/\mathbf{Z}I$–graded module on $Z^w_\chi(d\lambda)$ such that the generator $1 \otimes 1$ is homogeneous of degree $\lambda + \mathbf{Z}I$. We denote this graded module by $\widehat{Z}^w_\chi(\lambda)$. It is contained in $\mathcal{C}$.

Suppose that $w \in W^I$ and that α is a simple root with $ws_\alpha \in W^I$ and $w\alpha > 0$. We get then (for each $\lambda \in X$) homomorphisms (in $\mathcal{C}$)

$$\varphi : \widehat{Z}^w_\chi(\lambda) \to \widehat{Z}^{ws_\alpha}_\chi(\lambda - (p-1)w\alpha) \qquad \text{and} \qquad \varphi' : \widehat{Z}^{ws_\alpha}_\chi(\lambda - (p-1)w\alpha) \to \widehat{Z}^w_\chi(\lambda)$$

given by $\varphi(1 \otimes 1) = x^{p-1}_{w\alpha} \otimes 1$ and $\varphi'(1 \otimes 1) = x^{p-1}_{-w\alpha} \otimes 1$. Let r be the integer with $0 \le r < p$ and $\langle \lambda, w\alpha^\vee \rangle \equiv r \pmod p$. Then explicit calculations show: If $r = p-1$ then φ and φ' are isomorphisms. If $r < p-1$ one has $\ker(\varphi) = \operatorname{im}(\varphi')$ and $\ker(\varphi') = \operatorname{im}(\varphi)$. Furthermore $\ker(\varphi)$ is a homomorphic image of $\widehat{Z}^w_\chi(\lambda - (r+1)w\alpha)$. In both cases $\widehat{Z}^w_\chi(\lambda)$ and $\widehat{Z}^{ws_\alpha}_\chi(\lambda - (p-1)w\alpha)$ define the same class in the Grothendieck group of $\mathcal{C}$.

For all $w \in W$ and $\lambda \in X$ set $\lambda^w = \lambda - (p-1)(\rho - w\rho) \in X$. (Note that $\rho - w\rho \in \mathbf{Z}R$ even in case $\rho \notin X$.) The results from the last paragraph imply for all $w \in W^I$ and all $\lambda \in X$ that $\widehat{Z}^w_\chi(\lambda^w)$ and $\widehat{Z}^1_\chi(\lambda^1) = \widehat{Z}_\chi(\lambda)$ define the same class in the Grothendieck group of $\mathcal{C}$. (Use induction on the length of w. Note: If $w \in W^I$ and if α is a simple root with $w\alpha < 0$, then also $ws_\alpha \in W^I$.)

11.13. There exists a unique element $w^I \in W^I$ with $(w^I)^{-1}\beta < 0$ for all positive roots $\beta \notin \mathbf{Z}I$. We now have to know:

LEMMA. *For any $\lambda \in X$ the socle of $\widehat{Z}^{w^I}_\chi(\lambda^{w^I})$ is isomorphic to $\widehat{L}_\chi(\lambda)$.*

Assume this for the moment and return to Proposition 11.11. Fix $\lambda \in X$ and let n_β denote the integer with $\langle \lambda + \rho, \beta^\vee \rangle \equiv n_\beta \pmod p$ and $0 \le n_\beta < p$. Fix a reduced decomposition $w^I = s_1 s_2 \dots s_m$ and set $\sigma_i = s_1 s_2 \dots s_i$ for $0 \le i \le m$ (in particular $\sigma_0 = 1$). So $s_i = s_{\alpha_i}$ for some simple root α_i. The roots $\beta_i = \sigma_i \alpha_{i+1}$ with $0 \le i < m$ are distinct; they are precisely the positive roots β with $\beta \notin \mathbf{Z}I$.

The construction above yields a homomorphism $\varphi_i : \widehat{Z}^{\sigma_i}_\chi(\lambda^{\sigma_i}) \to \widehat{Z}^{\sigma_{i+1}}_\chi(\lambda^{\sigma_{i+1}})$ for each $i < m$. The composition of these φ_i is a homomorphism $\psi : \widehat{Z}_\chi(\lambda) \to \widehat{Z}^{w^I}_\chi(\lambda^{w^I})$. Since $\widehat{Z}_\chi(\lambda)$ has simple head isomorphic to $\widehat{L}_\chi(\lambda)$, and $\widehat{Z}^{w^I}_\chi(\lambda^{w^I})$ has simple socle isomorphic to $\widehat{L}_\chi(\lambda)$, the map ψ is either 0 or has kernel equal to the unique maximal submodule of $\widehat{Z}_\chi(\lambda)$. In particular, each composition factor $\widehat{L}_\chi(\mu)$ of $\widehat{Z}_\chi(\lambda)$ with $\mu + \mathbf{Z}I < \lambda + \mathbf{Z}I$ is a composition factor of $\ker(\psi)$, hence of some $\ker(\varphi_i)$. The analysis of φ and φ' above shows that φ_i is an isomorphism if $n_{\beta_i} = 0$. If $n_{\beta_i} > 0$, then each composition factor of $\ker(\varphi_i)$ is a composition factor of $\widehat{Z}^{\sigma_i}_\chi(\lambda^{\sigma_i} - n_{\beta_i}\beta_i)$, hence (by the result on classes in the Grothendieck group) one of $\widehat{Z}_\chi(\lambda - n_{\beta_i}\beta_i)$. Therefore each composition factor of $\ker(\psi)$ is a composition factor of some $\widehat{Z}_\chi(\lambda - n_\beta \beta)$ with $\beta > 0$, $\beta \notin \mathbf{Z}I$, and $n_\beta > 0$. Since $\lambda - n_\beta\beta = s_{\beta,rp} \bullet \lambda$ for a suitable $r \in \mathbf{Z}$ (depending on β) and since $\lambda - n_\beta\beta + \mathbf{Z}I < \lambda + \mathbf{Z}I$ we can now apply induction and get the proposition (modulo the lemma).

Remarks. a) The argument shows also that $\widehat{L}_\chi(\lambda)$ is not a composition factor of any $\ker(\varphi_i)$, hence not of $\ker(\psi)$. So ψ is non-zero and has image isomorphic to $\widehat{L}_\chi(\lambda)$. It follows that $\widehat{Z}_\chi(\lambda)$ is simple if and only if ψ is an isomorphism if and only if all φ_i are isomorphisms if and only if $n_\beta = 0$ for all $\beta > 0$, $\beta \notin \mathbf{Z}I$. If R is indecomposable, then one can check that this condition is equivalent to $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbf{Z}p$ for all $\alpha \in R$. (There are other ways of proving this under slightly more general conditions on p. It was proved for $G = \mathrm{GL}_n$ and $p > n$ in [**37**] and for general G in [**20**], Thm. 4.2. It was also announced in [**32**] and [**38**].)
b) Suppose that $\lambda, \mu \in C_I \cap X$. In order to get the 'strong' linkage principle one additional argument is needed: Let β be a positive root not in $\mathbf{Z}I$. Let $w \in W_{I,p}$ such that $w \bullet (\lambda - n_\beta\beta) \in C_I$. Then $w \bullet (\lambda - n_\beta\beta) \uparrow \lambda$. This follows from [**34**], 2.9.

11.14. Let us now turn to Lemma 11.13. It is proved by constructing a duality on $\mathcal{C}$ that fixes the simple modules and takes each $\widehat{Z}_\chi(\lambda)$ to $\widehat{Z}_\chi^{w^I}(\lambda^{w^I})$. Let us sketch how that is done.

The dual M^* of a $U_\chi(\mathfrak{g})$–module M is a $U_{-\chi}(\mathfrak{g})$–module. If M is in $\mathcal{C}$ then we give M^* an $X/\mathbf{Z}I$–grading such that each $(M^*)^\nu$ consists of all $f \in M^*$ with $f(M^{\nu'}) = 0$ for all $\nu' \neq -\nu$. Then M^* belongs to the analogue of $\mathcal{C}$, constructed with $-\chi$ instead of χ. One can now show for all $\lambda \in X$ that

$$\widehat{Z}_\chi(\lambda)^* \;\simeq\; \widehat{Z}_{-\chi}(-\lambda + (p-1)\,2\rho). \tag{1}$$

(For $\chi = 0$ this is [**29**], II.9.2(1). In order to extend to general χ one first has to check that [**29**], I.8.18 generalises. In a non-graded situation that can be found in [**16**], Cor. 1.2.b.)

We can similarly describe the dual of a baby Verma module over $\mathfrak{g}_I$. Let w_I be the unique element in W_I with $w_I(I) = -I$. Then $\rho - w_I\rho$ is the sum of all positive roots contained in $\mathbf{Z}I$, hence the analogue for $\mathfrak{g}_I$ of 2ρ. One has now for all $\lambda \in X$

$$Z_{\chi,I}(d\lambda)^* \;\simeq\; Z_{-\chi,I}(-d(\lambda + \rho - w_I\rho)).$$

Using

$$Z_{\chi,I}(d\lambda) \;\simeq\; Z_{\chi,I}(w_I \bullet d\lambda) \;=\; Z_{\chi,I}(d(w_I(\lambda+\rho) - \rho))$$

one can simplify the formula above to get

$$Z_{\chi,I}(d\lambda)^* \;\simeq\; Z_{-\chi,I}(-w_I d\lambda). \tag{2}$$

11.15. There is, by [**30**], 1.14, an automorphism τ of G (and hence of $\mathfrak{g}$) that stabilises T (and hence $\mathfrak{h}$), that satisfies $\chi \circ \tau^{-1} = -\chi$, and that induces $-w_I$ on X. For every $\mathfrak{g}$–module M let ${}^\tau M$ denote the $\mathfrak{g}$–module that coincides with M as a vector space and where each $x \in \mathfrak{g}$ acts on ${}^\tau M$ as $\tau^{-1}(x)$ does on M. The property $\chi \circ \tau^{-1} = -\chi$ implies: If M is a $U_{-\chi}(\mathfrak{g})$–module then ${}^\tau M$ is a $U_\chi(\mathfrak{g})$–module. If M is $X/\mathbf{Z}I$–graded then ${}^\tau M$ gets an $X/\mathbf{Z}I$–grading setting $({}^\tau M)^\nu = M^{-\nu}$ for all $\nu \in X/\mathbf{Z}I$. (Note that $\mu + \tau(\mu) = \mu - w_I(\mu) \in \mathbf{Z}I$ for all $\mu \in X$. Therefore τ acts as -1 on $X/\mathbf{Z}I$.) If M is in the analogue to $\mathcal{C}$ for $-\chi$, then ${}^\tau M$ is in $\mathcal{C}$.

Using that τ induces $-w_I$ on X one checks that $\tau(\mathfrak{n}^+) = w^I\mathfrak{n}^+$. If M is induced from a $U_{-\chi}(\mathfrak{b}^+)$–module V, then ${}^\tau M$ is induced from the $U_\chi(\tau\mathfrak{b}^+)$–module ${}^\tau V$ (defined in an obvious way). This yields easily for all $\lambda \in X$

$${}^\tau\widehat{Z}_{-\chi}(\lambda) \;\simeq\; \widehat{Z}_\chi^{w^I}(-w_I\lambda). \tag{1}$$

Furthermore $\tau_{|X} = -w_I$ implies also that $\tau(\mathfrak{g}_I) = \mathfrak{g}_I$. We get therefore also a functor $M \mapsto {}^\tau M$ taking $U_{-\chi}(\mathfrak{g}_I)$–modules to $U_\chi(\mathfrak{g}_I)$–modules. Using $\tau(\mathfrak{g}_I \cap \mathfrak{n}^+) = \mathfrak{g}_I \cap \mathfrak{n}^+$ one gets arguing as above for all $\lambda \in X$

$${}^\tau Z_{-\chi,I}(d\lambda) \;\simeq\; Z_{\chi,I}(-w_I d\lambda). \tag{2}$$

11.16. Consider now the composition $M \mapsto {}^\tau(M^*)$ of these two functors. This is a duality on the category of all $U_\chi(\mathfrak{g})$–modules, on $\mathcal{C}$, and on the category of all $U_\chi(\mathfrak{g}_I)$–modules. Our previous isomorphisms yield for all $\lambda \in X$

$$\begin{aligned}{}^\tau(\widehat{Z}_\chi(\lambda)^*) &\simeq {}^\tau(\widehat{Z}_\chi(w_I \bullet \lambda)^*) \simeq {}^\tau\widehat{Z}_{-\chi}(-w_I \bullet \lambda + 2(p-1)\rho)\\ &\simeq \widehat{Z}_\chi^{w^I}(\lambda + \rho - w_I\rho - 2(p-1)w_I\rho)\\ &\simeq \widehat{Z}_\chi^{w^I}(\lambda + \rho + w_I\rho)\,\langle -2pw_I\rho\rangle.\end{aligned}$$

Now use that $w_I\rho = -w^I\rho$ and $\rho - w_I\rho \in \mathbf{Z}I$, hence that $-2pw_I\rho + \mathbf{Z}I = -p(\rho - w^I\rho) + \mathbf{Z}I$ to get

$${}^\tau(\widehat{Z}_\chi(\lambda)^*) \;\simeq\; \widehat{Z}_\chi^{w^I}(\lambda^{w^I}). \tag{1}$$

Furthermore our formulas for the $\mathfrak{g}_I$–modules yield

$$^\tau(Z_{\chi,I}(d\lambda)^*) \;\simeq\; Z_{\chi,I}(d\lambda). \tag{2}$$

Looking at the construction one checks for all M in $\mathcal{C}$ that each $(^\tau(M^*))^\nu$ is isomorphic to $^\tau((M^\nu)^*)$ as a $\mathfrak{g}_I$–module. This implies for all $\lambda \in X$ that $\lambda + \mathbf{Z}I$ is the largest degree in the grading of $^\tau(\widehat{L}_\chi(\lambda)^*)$ and that the graded piece of degree $\lambda + \mathbf{Z}I$ in $^\tau(\widehat{L}_\chi(\lambda)^*)$ is isomorphic to $^\tau(Z_{\chi,I}(d\lambda)^*)$, hence by (2) to $Z_{\chi,I}(d\lambda)$. Since $^\tau(\widehat{L}_\chi(\lambda)^*)$ has to be simple, the classification of simple modules in $\mathcal{C}$ implies (for all $\lambda \in X$)

$$^\tau(\widehat{L}_\chi(\lambda)^*) \;\simeq\; \widehat{L}_\chi(\lambda). \tag{3}$$

Because a duality takes a head to a socle, the formulas (1) and (3) imply Lemma 11.13.

11.17. Let us look at another consequence of 11.16(3). If N is a $U_{-\chi}(\mathfrak{g})$–module, then we have for all simple $U_\chi(\mathfrak{g})$–modules L

$$[N^* : L] \;=\; [N : L^*] \;=\; [^\tau N : {}^\tau(L^*)] \;=\; [^\tau N : L].$$

So N^* and $^\tau N$ define the same class in the Grothendieck group of all $U_\chi(\mathfrak{g})$–modules. Similarly, if N is in the analogue to $\mathcal{C}$ for $-\chi$, then N^* and $^\tau N$ define the same class in the Grothendieck group of $\mathcal{C}$.

Recall the functors $\mathcal{Z}$ and $\mathcal{Z}'$ (from $U_\chi(\mathfrak{g}_I)$–modules to $U_\chi(\mathfrak{g})$–modules) introduced before Proposition 10.7. They can be defined similarly on $U_{-\chi}(\mathfrak{g}_I)$–modules (taking them to $U_{-\chi}(\mathfrak{g})$–modules). We have $\tau(\mathfrak{u}) = \mathfrak{u}'$, hence $\tau(\mathfrak{p}) = \mathfrak{p}'$. Arguing as before one gets now for every $U_{-\chi}(\mathfrak{g}_I)$–module V'

$$^\tau\mathcal{Z}'(V') \;\simeq\; \mathcal{Z}(^\tau V'),$$

hence for every $U_\chi(\mathfrak{g}_I)$–module V

$$^\tau\mathcal{Z}'(V^*) \;\simeq\; \mathcal{Z}(^\tau(V^*)).$$

Any simple $U_\chi(\mathfrak{g}_I)$–module E (isomorphic to some $Z_{\chi,I}(d\lambda)$) satisfies $^\tau(E^*) \simeq E$ by 11.16(2). This implies that

$$^\tau\mathcal{Z}'(E^*) \;\simeq\; \mathcal{Z}(E). \tag{1}$$

The remark in the preceding paragraph shows now that $\mathcal{Z}'(E^*)^*$ and $\mathcal{Z}(E)$ define the same class in the Grothendieck group of of all $U_\chi(\mathfrak{g})$–modules. This is required to get the last part of Proposition 10.11.

11.18. One gets also a graded version of Proposition 10.11. Given $\lambda \in X$ we give $Q^I_\chi(d\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} Q_{\chi,I}(d\lambda)$ a grading such that $1 \otimes Q_{\chi,I}(d\lambda)$ is homogeneous of degree $\lambda + \mathbf{Z}I$. Denote this graded module by $\widehat{Q}^I_\chi(\lambda)$. It belongs to $\mathcal{C}$. It has a filtration of length $W_I \bullet d\lambda$ where each quotient of subsequent terms in the filtration is isomorphic to $\widehat{Z}_\chi(\lambda)$.

Using graded versions of the formulas above and of Nakano's arguments from [**36**] one can show:

Proposition. *For each $\lambda \in X$ let $\widehat{Q}_\chi(\lambda)$ denote the projective cover of $\widehat{L}_\chi(\lambda)$ in $\mathcal{C}$. Then $\mathcal{F}(\widehat{Q}_\chi(\lambda)) \simeq Q_\chi(d\lambda)$. Furthermore $\widehat{Q}_\chi(\lambda)$ has a filtration where each quotient of subsequent terms in the filtration is isomorphic to some $\widehat{Q}^I_\chi(\mu)$. The number of factors isomorphic to a given $\widehat{Q}^I_\chi(\mu)$ is equal to $[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]$.*

11.19. Propositions 11.18 and 11.11 imply that any composition factor $\widehat{L}_\chi(\lambda')$ of $\widehat{Q}_\chi(\lambda)$ satisfies $\lambda' \in W_p \bullet \lambda$. It follows for each indecomposable M in $\mathcal{C}$ that we have $\lambda' \in W_p \bullet \lambda$ for all composition factors $\widehat{L}_\chi(\lambda)$ and $\widehat{L}_\chi(\lambda')$ of M.

Set

$$C_0 = \{ \lambda \in X_{\mathbf{R}} \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p \quad \forall \alpha \in R^+ \}. \tag{1}$$

This is a fundamental domain for the dot action of W_p on $X_{\mathbf{R}}$. Therefore $C_0 \cap X$ parametrises the orbits of W_p on X. Set

$$W^{I,p} \;=\; \{ \sigma \in W_p \mid \sigma \bullet C_0 \subset C_I \}. \tag{2}$$

Then C_I is the union of the $\sigma \bullet C_0$ with $\sigma \in W^{I,p}$. If $\lambda \in C_0$, then $W_p \bullet \lambda \cap C_I = W^{I,p} \bullet \lambda$.

The remarks above imply now that each M in $\mathcal{C}$ has a direct sum decomposition

$$M \;=\; \bigoplus_{\mu \in C_0 \cap X} \operatorname{pr}_\mu(M) \tag{3}$$

such that for each μ all composition factors of $\operatorname{pr}_\mu(M)$ have the form $\widehat{L}_\chi(\lambda)$ with $\lambda \in W_p \bullet \mu$.

Let $\mathcal{C}(\mu)$ denote the subcategory of all M in $\mathcal{C}$ with $M = \operatorname{pr}_\mu(M)$. Then $\mathcal{C}$ is the direct product of all $\mathcal{C}(\mu)$ with $\mu \in C_0 \cap X$.

11.20. We can define for all $\lambda, \mu \in C_0 \cap X$ a *translation functor* T_λ^μ from $\mathcal{C}(\lambda)$ to $\mathcal{C}(\mu)$ as follows: Take the simple G–module E with highest weight in $W(\mu - \lambda)$. Considered as a $\mathfrak{g}$–module it has p–character 0. Therefore $M \mapsto E \otimes M$ takes $U_\chi(\mathfrak{g})$–modules to $U_\chi(\mathfrak{g})$–modules. We can give E an $X/\mathbf{Z}I$–grading such that $E^{\nu+\mathbf{Z}I}$ is the direct sum of all T–weight spaces in E with weights in $\nu + \mathbf{Z}I$. For M in $\mathcal{C}$ we give $E \otimes M$ the natural grading of a tensor product; then $E \otimes M$ is again in $\mathcal{C}$. Now define

$$T_\lambda^\mu(M) \;=\; \operatorname{pr}_\mu(E \otimes M)$$

for all M in $\mathcal{C}(\lambda)$.

Now standard results on translation functors generalise to our present situation. The first thing to observe is that each $E \otimes \widehat{Z}_\chi(\nu)$ has a filtration with factors $\widehat{Z}_\chi(\nu+\nu')$ with ν' running over the weights of E counted with their multiplicities. More generally, each $E \otimes \widehat{Z}_\chi^w(\nu)$ with $w \in W^I$ has a filtration with factors $\widehat{Z}_\chi^w(\nu + \nu')$ and ν' as before.

11.21. Suppose that μ is in the closure of the facet of λ (see [**29**], II.6.2). One gets now for all $\sigma \in W_p$ that

$$T_\lambda^\mu \, \widehat{Z}_\chi(\sigma \bullet \lambda) \;\simeq\; \widehat{Z}_\chi(\sigma \bullet \mu) \tag{1}$$

and, more generally, for all $w \in W^I$ that $T_\lambda^\mu \, \widehat{Z}_\chi^w((\sigma \bullet \lambda)^w) \simeq \widehat{Z}_\chi^w((\sigma \bullet \mu)^w)$, cf. [**1**], 7.11.

Let $\sigma \in W^{I,p}$. The simple module $\widehat{L}_\chi(\sigma \bullet \lambda)$ is the image of a homomorphism $\widehat{Z}_\chi(\sigma \bullet \lambda) \to \widehat{Z}_\chi^{w^I}((\sigma \bullet \lambda)^{w^I})$. Therefore the exactness of T_λ^μ implies that $T_\lambda^\mu \widehat{L}_\chi(\sigma \bullet \lambda)$ is the image of a homomorphism $\widehat{Z}_\chi(\sigma \bullet \mu) \to \widehat{Z}_\chi^{w^I}((\sigma \bullet \mu)^{w^I})$, hence either 0 or isomorphic to $\widehat{L}_\chi(\sigma \bullet \mu)$, cf. [**29**], II.7.14 or [**1**], 7.13. Furthermore, since $\widehat{L}_\chi(\sigma \bullet \mu)$ is a composition factor of $\widehat{Z}_\chi(\sigma \bullet \mu) \simeq T_\lambda^\mu \widehat{Z}_\chi(\sigma \bullet \lambda)$, there has to exist a composition factor $\widehat{L}_\chi(\sigma' \bullet \lambda)$ of $\widehat{Z}_\chi(\sigma \bullet \lambda)$ with $\sigma' \in W^{I,p}$ and $T_\lambda^\mu \widehat{L}_\chi(\sigma' \bullet \lambda) \simeq \widehat{L}_\chi(\sigma \bullet \mu)$. We know already that $T_\lambda^\mu \widehat{L}_\chi(\sigma' \bullet \lambda)$ is either 0 or isomorphic to $\widehat{L}_\chi(\sigma' \bullet \mu)$. Since $\sigma' \bullet \mu \in C_I$, this implies that $\sigma' \bullet \mu = \sigma \bullet \mu$. The precise determination of $\sigma' \bullet \lambda$ requires extra work except in one case: If λ and μ have the same stabiliser in W_p then the last equality yields $\sigma' \bullet \lambda = \sigma \bullet \lambda$, hence:

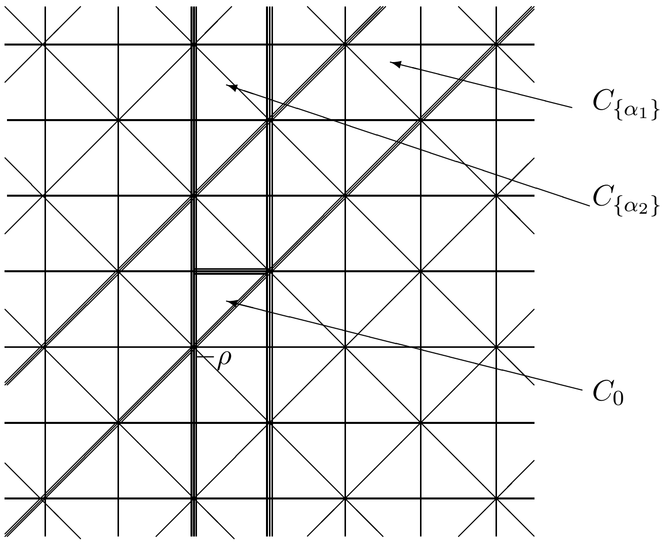
Proposition. *Suppose that $\lambda, \mu \in C_0 \cap X$ belong to the same facet with respect to W_p. Then*

$$T_\lambda^\mu \widehat{L}_\chi(\sigma \bullet \lambda) \simeq \widehat{L}_\chi(\sigma \bullet \mu)$$

for all $\sigma \in W_p$.

In fact, in this situation T_λ^μ is an equivalence of categories between $\mathcal{C}(\lambda)$ and $\mathcal{C}(\mu)$.

11.22. Example 4. Suppose $\mathfrak{g}$ is of type B_2 with the simple roots $\{\alpha_1, \alpha_2\}$ such that α_1 is long. Then the following diagram illustrates the alcoves and I-alcoves.

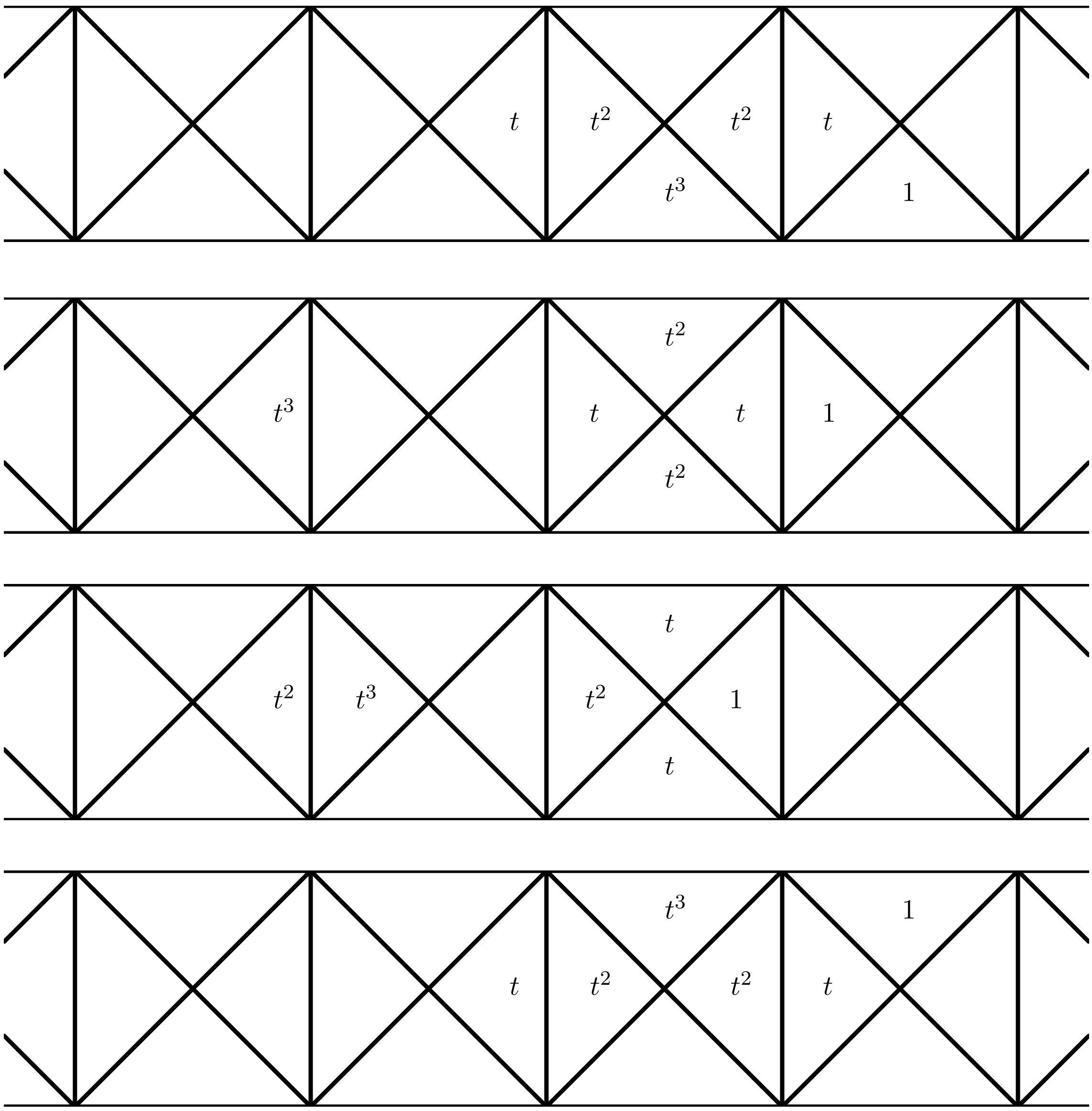


The diagram shows (a part of) the plane $X_{\mathbf{R}}$. The lines are the reflection hyperplanes (with equations of the form $\langle x+\rho, \alpha^\vee \rangle = rp$ with $\alpha \in R$ and $r \in \mathbf{Z}$). The small triangles formed by these lines are the alcoves with respect to W_p; one of these alcoves is C_0. For $I = \{\beta\}$ with $\beta \in \{\alpha_1, \alpha_2\}$ the set C_I is given by the condition $0 \le \langle x+\rho, \beta^\vee \rangle \le p$. It is bounded by two parallel lines; these are drawn thicker in the diagram. The other two C_I are $C_\emptyset = X_{\mathbf{R}}$ and $C_{\{\alpha_1,\alpha_2\}} = C_0$.

Choose a weight $\lambda_0 \in X$ that is contained in the interior of C_0, i.e., that satisfies $0 < \langle \lambda_0 + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$. Then the structure of the baby Verma modules in $\mathcal{C}(\lambda_0)$ can be described (to some extent) by the polynomials (in one variable t)

$$F_{v,w} = \sum_{i \ge 0} [\operatorname{rad}^i \widehat{Z}_\chi(v \bullet \lambda_0) / \operatorname{rad}^{i+1} \widehat{Z}_\chi(v \bullet \lambda_0) : \widehat{L}_\chi(w \bullet \lambda_0)]\, t^i \qquad (1)$$

for all $v, w \in W^{I,p}$. By the translation principle these polynomials are independent of the choice of λ_0. For $I = \{\alpha_1\}$ the Loewy series of all $\widehat{Z}_\chi(\lambda)$ were determined in [**31**]. The results in [**31**], Thm. 3.12 can be translated into formulas for the $F_{v,w}$. These results are illustrated by the diagrams below. They show C_I for $I = \{\alpha_1\}$ (rotated) and the alcoves contained in C_I. We fix some $v \in W^{I,p}$ and write $F_{v,w}$ into the alcove $w \bullet C_0$ (for all $w \in W^{I,p}$ with $F_{v,w} \neq 0$). The alcove $v \bullet C_0$ can be read off the diagram since $F_{v,v} = 1$ while all other $F_{v,w}$ are divisible by t.



11.23. Besides this example there are only few cases where the dimensions of the simple modules and the composition factors (with their multiplicities) of the baby Verma modules are explicitly known. The case where I consists of all simple roots is of course taken care of by Proposition 10.5. At the other extreme, for $I = \emptyset$ Curtis's theorem (see 10.4) reduces the problem to an analogous one for G. Here the answer is known for groups of rank up to 2 and for type A_3 for all primes, while for arbitrary G the answer for large p (greater than an unknown bound) is given by Lusztig's conjecture.

In the case where R is of type A_2 and $|I| = 1$ the simple modules are described in [**32**], see also [**20**], 3.6. In [**30**] the cases are treated where R is of type A_n and I defines a subsystem of type A_{n-1}, and where R is of type B_n and I defines a subsystem of type B_{n-1} (These are the two cases where one can find χ in the 'subregular nilpotent orbit' that has standard Levi form.) The remaining case for R of type B_2 is Example 4 above.

11.24. For all $v, w \in W^{I,p}$ Lusztig, [**34**], has constructed a polynomial $P_{v,w} \in \mathbf{Z}[t^{-1}]$ generalising the classical Kazhdan-Lusztig polynomials. Another approach to these polynomials can be found in Soergel's *Appendix zu "Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln"* available from http://sun2.mathematik.uni-freiburg.de/home/soergel.

The element $w_I \in W_I$ with $w_I(I) = -I$ satisfies $-w_I(C_I + \rho) = C_I + \rho$. This implies that there exists for each $w \in W^{I,p}$ a unique element $\kappa_I(w) \in W^{I,p}$ with $\kappa_I(w)(C_0 + \rho) = -w_I w(C_0 + \rho)$. The map κ_I is an involution on $W^{I,p}$.

Lusztig's Hope ([**34**], 13.17): If λ_0 is a weight in the interior of C_0 then

$$[\widehat{Z}_\chi(v \bullet \lambda_0) : \widehat{L}_\chi(w \bullet \lambda_0)] = P_{\kappa_I(v),\kappa_I(w)}(1).$$

Moreover one may also hope that $P_{\kappa_I(v),\kappa_I(w)} = F_{v,w}(t^{-1})$ where $F_{v,w}$ is defined as in 11.22(1).

If $\chi = 0$ then these hopes reduce to the Lusztig conjecture. For χ regular nilpotent everything becomes trivial. The results in [**30**] and [**31**] show that the hopes are true in the cases considered there.

11.25. One says that $\chi \in \mathfrak{g}^*$ is subregular nilpotent if it is nilpotent and if $\dim(\mathfrak{c}_\mathfrak{g}(\chi)) = \dim(\mathfrak{h}) + 2$. If R is of type A_r or B_r then we can find χ that is subregular nilpotent and has standard Levi form. Choose in this case a weight λ_0 in the interior of C_0 and let α_0 be the largest short root in R. Write

$$\alpha_0^\vee = \sum_{i=1}^{r} n_i \alpha_i{}^\vee,$$

where the simple roots of R are $\{\alpha_1, \dots, \alpha_r\}$. The results in [**30**] show that there are $1 + \sum_i n_i$ simple $U_\chi(\mathfrak{g})$–modules with the same central characters as $Z_\chi(\lambda_0)$. Out of these n_i have dimension $\langle \lambda + \rho, \alpha_i^\vee \rangle p^{N-1}$ (for each i) and one has dimension $(p - \sum_{i=1}^r n_i \langle \lambda + \rho, \alpha_i^\vee \rangle) p^{N-1} = (p - \langle \lambda + \rho, \alpha_0^\vee \rangle) p^{N-1}$ where $N = |R^+|$. They all occur with multiplicity 1 in $Z_\chi(\lambda_0)$.

One may wonder whether this behaviour generalises to subregular nilpotent χ for other types.

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PS. (2003) This survey appeared in the Proceedings of the NATO Advanced Study Institute on Representation Theories and Algebraic Geometry, Montreal, 28 July – 8 August 1997. I have now corrected a few typos and updated two references. I did not correct the second line in 9.5 where "such that $Z_\chi(\lambda)$ is defined" should be replaced by "such that $\chi(\mathfrak{n}^-) = \chi(\mathfrak{n}^+) = 0$." As far as the last sentence in 11.25 is concerned: My expectation there has turned out to be not quite true, see my paper in *Represent. Theory* **3** (1999), 153–222